# Uttar Pradesh Higher Education Service Commission
# Assistant Professor 2021
## Solved Paper [Exam Date : 28.11.2021]

**1. Solution of partial differential equation $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ ,where $u(x,0) = 6e^{-3x}$ is**

**अंशिक अवकलन समीकरण $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ का हल जबकि $u(x,0) = 6e^{-3x}$ है-**

(a) $6e^{-(2x+3t)}$ (b) $6e^{-(3x+4t)}$
(c) $6e^{(3x+4t)}$ (d) $6e^{-(3x+2t)}$

**Ans. (d) :** $6e^{-(3x+2t)}$

Given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

Let u = X(x)T(t), where X is a function of x only and T is a function of t only.

Now given equation can be written as,

$$\frac{\partial}{\partial x}\left(X(x)T(t)\right) = 2\frac{\partial}{\partial t}\left(X(x)T(t)\right) + X(x)T(t)$$

$$\text{or } T(t)\frac{d}{dx}\left(X(x)\right) = 2X(x)\frac{d}{dt}\left(T(t)\right) + X(x)T(t)$$

On separating the variables we have,

$$\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1 = C$$

$$\Rightarrow \frac{1}{X}\frac{dX}{dx} = C \;\&\; \frac{2}{T}\frac{dT}{dt} + 1 = C$$

$$\Rightarrow \frac{dX}{dx} = CX \;\&\; \frac{dT}{dt} + \frac{T}{2} = \frac{CT}{2}$$

$$\Rightarrow DX - CX = 0 \;\&\; DT - \left(\frac{C}{2} - \frac{1}{2}\right)T = 0$$

$$\Rightarrow \text{A.E. is } m - C = 0 \;\&\; \text{A.E. is } m - \left(\frac{C}{2} - \frac{1}{2}\right) = 0$$

$$\Rightarrow m = C \qquad \Rightarrow m = \frac{1}{2}(C-1)$$

$$\Rightarrow X = ae^{Cx} \qquad \Rightarrow T = be^{\frac{1}{2}(C-1)t}$$

Then we get from u = X(x)T(t)

$$u = ae^{Cx}\, be^{\frac{1}{2}(C-1)t}$$

$$\Rightarrow u = abe^{Cx + \frac{1}{2}(C-1)t}$$

Because $u(x, 0) = 6e^{-3x}$ we have from above

$$6e^{-3x} = abe^{Cx}$$

$$\Rightarrow \quad ab = 6 \;\&\; C = -3$$

Thus $u = 6e^{-3x + \frac{1}{2}(-3-1)t}$

$\Rightarrow$ $u = 6e^{-3x-2t}$ , which is the required solution.

**2. Transformation of Laplace differential equation $\left(\frac{\partial^2 \nu}{\partial x^2}\right) + \left(\frac{\partial^2 \nu}{\partial y^2}\right) = 0$ into polar form r, θ in $R^2$. where $\nu \equiv \nu(x, y)$, is**

**लाप्लास अवकल समीकरण $\left(\frac{\partial^2 \nu}{\partial x^2}\right) + \left(\frac{\partial^2 \nu}{\partial y^2}\right) = 0$ का रूपांतरण ध्रुवीय रूप r, θ में (जो कि $R^2$ में है) जबकि $\nu \equiv \nu(x, y)$.**

(a) $\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r}\frac{\partial \nu}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \nu}{\partial \theta^2} = 0$

(b) $\frac{\partial^2 \nu}{\partial r^2} - \frac{1}{r}\frac{\partial \nu}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \nu}{\partial \theta^2} = 0$

(c) $\frac{\partial^2 \nu}{\partial r^2} - \frac{1}{r}\frac{\partial \nu}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \nu}{\partial \theta^2} = 0$

(d) $\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r}\frac{\partial \nu}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \nu}{\partial \theta^2} = 0$

**Ans. (a) :** $\frac{\partial^2 \nu}{\partial r^2} + \frac{1}{r}\frac{\partial \nu}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \nu}{\partial \theta^2} = 0$

Laplace's equation is given by

$$\left(\frac{\partial^2 \nu}{\partial x^2}\right) + \left(\frac{\partial^2 \nu}{\partial y^2}\right) = 0$$

Here x, y are cartesian coordinates in plane.

The expression $\left(\frac{\partial^2 \nu}{\partial x^2}\right) + \left(\frac{\partial^2 \nu}{\partial y^2}\right)$ is called Laplacian of ν.

Laplacian of $\nu \equiv \nu(x, y)$ in polar coordinates r, θ defined by x = r cos θ, y = r sin θ; thus

$$r = \sqrt{x^2 + y^2},\ \tan\theta = \frac{y}{x}$$

By the chain rule we obtain

$$\nu_x = \nu_r\, r_x + \nu_\theta\, \theta_x$$

Subscripts denoting partial derivatives.

Differentiating once more with respect to x gives

$\nu_{xx} = (\nu_r r_x)_x + (\nu_\theta \theta_x)_x$

$= (\nu_r)_x r_x + \nu_r r_{xx} + (\nu_\theta)_x \theta_x + \nu_\theta \theta_{xx}$

$= (\nu_{rr} r_x + \nu_{r\theta}\theta_x)\, r_x + \nu_r\, r_{xx} + (\nu_{\theta r} r_x + \nu_{\theta\theta}\theta_x)\theta_x + \nu_\theta \theta_{xx}$ ...(1)

Also, by differentiation of r and θ we find

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r},\ \theta_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}$$

Differentiating again we obtain

$$r_{xx}=\frac{r-xr_x}{r^2}=\frac{1}{r}-\frac{x^2}{r^3}=\frac{y^2}{r^3}, \theta_{xx}=-y\left(-\frac{2}{r^3}\right)r_x=\frac{2xy}{r^4}$$

Substituting all these expressions in the equation (1). Assuming continuity of the first and second partial derivatives we have $\nu_{r\theta}=\nu_{\theta r}$ and by simplifying

$$\nu_{xx}=\frac{x^2}{r^2}\nu_{rr}-2\frac{xy}{r^3}\nu_{r\theta}+\frac{y^2}{r^4}\nu_{\theta\theta}+\frac{y^2}{r^3}\nu_r+2\frac{xy}{r^4}\nu_\theta$$

Similarly

$$\nu_{yy}=\frac{y^2}{r^2}\nu_{rr}+2\frac{xy}{r^3}\nu_{r\theta}+\frac{x^2}{r^4}\nu_{\theta\theta}+\frac{x^2}{r^3}\nu r-2\frac{xy}{r^4}\nu_\theta$$

By adding we obtain the Laplacian of $\nu \equiv \nu(x,y)$ in polar coordinates

$$\frac{\partial^2\nu}{\partial r^2}+\frac{1}{r}\frac{\partial\nu}{\partial r}+\frac{1}{r^2}\frac{\partial^2\nu}{\partial\theta^2}$$

Thus, Laplace's equation $\left(\frac{\partial^2\nu}{\partial x^2}\right)+\left(\frac{\partial^2\nu}{\partial y^2}\right)=0$ into polar coordinates is

$$\frac{\partial^2\nu}{\partial r^2}+\frac{1}{2}\frac{\partial\nu}{\partial r}+\frac{1}{r^2}\frac{\partial^2\nu}{\partial\theta^2}=0$$

**3. The solution of the Volterra integral equation $y(x)=1+\int_0^x(x-t)y(t)dt$, when $y_0(x)=0$, is**

**वोल्टेरा समाकल समीकरण $y(x)=1+\int_0^x(x-t)y(t)dt$, जबकि $y_0(x)=0$ का हल है-**

(a) y (x) = sinh x (b) y(x) = cos x
(c) y(x) = sin x (d) y(x) = cosh x

**Ans. (d) :** y(x) = coshx

Given $y(x)=1+\int_0^x(x-t)y(t)dt$ and $y_0(x)=0$

On comparing with $y(x)=f(x)+\lambda\int_0^x k(x,t)y(t)dt$ we obtain

$$f(x)=1,\ \lambda=1,\ K(x,t)=x-t$$

The n[th] order approximation is given by

$$y_n(x)=f(x)+\lambda\int_0^x k(x,t)y_{n-1}(t)dt$$

or $$y_n(x)=1+\int_0^x(x-t)y_{n-1}(t)dt$$

Now n=1 gives

$$y_1(x)=1+\int_0^x(x-t)y_0(t)dt=1$$

n = 2 gives

$$y_2(x)=1+\int_0^x(x-t)y_1(t)dt$$

$$=1+\int_0^x(x-t)dt$$

$$=1+\left[xt-\frac{t^2}{2}\right]_0^x=1+x^2-\frac{x^2}{2}=1+\frac{x^2}{2}$$

n = 3 gives

$$y_3(x)=1+\int_0^x(x-t)y_2(t)dt=1+\int_0^x(x-t)\left(1+\frac{t^2}{2}\right)dt$$

$$=1+\int_0^x\left(x+\frac{xt^2}{2}-t-\frac{t^3}{2}\right)dt$$

$$=1+\left[xt+\frac{xt^3}{6}-\frac{t^2}{2}-\frac{t^4}{8}\right]_0^x$$

$$=1+x^2+\frac{x^4}{6}-\frac{x^2}{2}-\frac{x^4}{8}=1+\frac{x^2}{2}+\frac{x^4}{24}$$

$$=1+\frac{x^2}{2!}+\frac{x^4}{4!}$$

Similarly we can write

$$y_n(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+.....+\frac{x^{2n-2}}{(2n-2)!}$$

Now the required solution y(x) is given by

$$y(x)=\lim_{n\to\infty}y_n(x)$$

Which gives y(x) = coshx

**4. Laplace transform of Bessel function $J_1(x)$ is -**

**बैसल फलन $J_1(x)$ का लाप्लास रूपांतरण होगा-**

(a) $1+\frac{s}{\sqrt{s^2+1}}$ (b) $1+\frac{1}{(s^2+1)^{3/2}}$

(c) $1+\frac{s}{(s^2+1)^{3/2}}$ (d) $1-\frac{s}{\sqrt{s^2+1}}$

**Ans. (d) :** $1-\frac{s}{\sqrt{s^2+1}}$

Bessel function of the first kind of order n,

$$J_n(x)=x^n\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m+n}m!(n+m)!}$$

For n = 0 we obtain the bessel function of order 0

$$J_0(x)=\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

$$=1-\frac{x^2}{2^2(1!)^2}+\frac{x^4}{2^4(2!)^2}-\frac{x^6}{2^6(3!)^2}+.....$$

$$=1-\frac{x^2}{2^2}+\frac{x^4}{2^24^2}-\frac{x^6}{2^24^26^2}+.....$$

Then Laplace transform of $J_0(x)$ is

$$\mathcal{L}\{J_0(x)\}=\frac{1}{s}-\frac{1}{2^2}.\frac{2!}{s^3}+\frac{1}{2^24^2}.\frac{4!}{s^5}-\frac{1}{2^24^26^2}\frac{6!}{s^7}+.....$$

$$=\frac{1}{s}\left\{1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1.3}{2.4}\left(\frac{1}{s^4}\right)-\frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right)+.....\right\}$$

$$=\frac{1}{s}\left\{\left(1+\frac{1}{s^2}\right)^{-1/2}\right\}=\frac{1}{\sqrt{s^2+1}}$$

using binomial theorem,

Now we have $J_0'(x) = -J_1(x)$. Hence Laplace transform of $J_1(x)$ is

$$\mathcal{L}\{J_1(x)\} = -\mathcal{L}\{J_0'(x)\} = -[s\,\mathcal{L}\{J_0(x)\} - 1]$$

$$= 1 - \frac{s}{\sqrt{s^2+1}}$$

**5. Inverse Laplace transform of $\frac{e^{-4s}}{(s+2)^3}$ is —**

$\frac{e^{-4s}}{(s+2)^3}$ **का व्युत्क्रम लाप्लास रूपांतरण है –**

(a) $\frac{1}{2}(x-4)^2 e^{-2(x-4)} H(x-4)$

(b) $\frac{1}{2}(x-4)^2 e^{-2(x-4)} H(x+4)$

(c) $(x-4)^2 e^{-2(x-4)}$

(d) $(x-4)^2 e^{-(x-4)}$

**Ans. (a) :** $\frac{1}{2}(x-4)^2 e^{-2(x-4)} H(x-4)$

Inverse Laplace transform of $\frac{1}{(s+2)^3}$ i.e.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = e^{-2x}\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{x^2 e^{-2x}}{2!} = \frac{1}{2}x^2 e^{-2x}$$

Now if inverse Laplace transform of f(s) is F(x) i.e. $\mathcal{L}^{-1}\{f(s)\} = F(x)$ then inverse Laplace transform of $e^{-as}f(s)$ i.e. $\mathcal{L}^{-1}\{e^{-as}f(s)\} = G(x)$ where,

$$G(x) = \begin{cases} F(x-a), & x > a \\ 0, & x < a \end{cases}$$

then $$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{(s+2)^3}\right\} = \begin{cases} \frac{1}{2}(x-4)^2 e^{-2(x-4)}, & x > 4 \\ 0, & x < 4 \end{cases}$$

$$= \frac{1}{2}(x-4)^2 e^{-2(x-4)} H(x-4)$$

**6. The solution of the Fredholm linear integral equation**

$u(x) + \int_0^1 x(e^{xt}-1)u(t)dt = e^x - x$ **is**

**फ्रेडहोल्म की रैखिक समाकल समीकरण** $u(x) + \int_0^1 x(e^{xt}-1)u(t)dt = e^x - x$ **का हल है -**

(a) $u(x) = 1$ (b) $u(x) = e^x - x$

(c) $u(x) = 3$ (d) $u(x) = e^x$

**Ans. (a) :** u(x) = 1

Given, $u(x) + \int_0^1 x(e^{xt}-1)u(t)dt = e^x - x$

Let u(x) = 1 then given equation becomes

$$1 + \int_0^1 x(e^{xt}-1)dt = e^x - x$$

$$\text{L.H.S.} = 1 + \left[x\frac{e^{xt}}{x} - xt\right]_0^1 = 1 + e^x - x - 1 = e^x - x = \text{R.H.S.}$$

Hence the exact solution of the given integral equation is

u(x) = 1

**7. The Homogeneous integral equation**

$\phi(x) - \lambda\int_0^1 (3x-2)t\phi(t)dt = 0$ **has**

**समघात समाकल समीकरण की,**

(a) One characteristic number.
एक अभिलक्षणी संख्या है।

(b) Two characteristic number.
दो अभिलक्षणी संख्या है।

(c) Three characteristic number
तीन अभिलक्षणी संख्या है।

(d) No characteristic number
कोई अभिलक्षणी संख्या नहीं है।

**Ans. (d) :** no characteristic number

Given, $\phi(x) = \lambda\int_0^1 (3x-2)t\phi(t)dt$

or $\phi(x) = \lambda(3x-2)\int_0^1 t\phi(t)dt$

Let $C = \int_0^1 t\phi(t)dt$

Then $\phi(x) = \lambda C(3x-2)$

and $\phi(t) = \lambda C(3t-2)$

Hence $C = \int_0^1 \lambda Ct(3t-2)dt$

or $C = \lambda C\left[t^3 - t^2\right]_0^1$

or C = 0

Thus $\phi(x) = 0$, which is zero solution. Hence for any $\lambda$, equation has only zero solution $\phi(x) = 0$. Therefore, equation does not possess any characteristic number or eigenfunction.

**8. The Resolvent Kernel R (x, t; λ) of the Volterra integral equation** $y(x) = 1 + \lambda\int_0^x e^{3(x-t)}y(t)dt$ **shall be,**

**वोल्टेरा समाकल समीकरण** $y(x) = 1 + \lambda\int_0^x e^{3(x-t)}y(t)dt$ **का विघटक अष्टि R (x, t; λ) होगी-**

(a) $e^{(3-\lambda)(x-t)}$ (b) $e^{(3-\lambda)(x+t)}$

(c) $e^{(3+\lambda)(x+t)}$ (d) $e^{(3+\lambda)(x-t)}$

**Ans. (d) :** $e^{(3+\lambda)(x-t)}$

The kernel $k(x,t) = e^{3(x-t)}$

Iterated kernels $k_n(x,t)$ are given by

$k_1(x,t) = k(x,t)$

$k_n(x,t) = \int_t^x k(x,z)k_{n-1}(z,t)dz, n = 2,3......$

$\therefore$ $k_1(x,t) = e^{3(x-t)}$

& n= 2 gives

$$k_2(x,t) = \int_t^x k(x,z)k_1(z,t)dz$$

$$= \int_t^x e^{3(x-z)}e^{3(z-t)}dz = e^{3(x-t)}\int_t^x dz = e^{3(x-t)}(x-t)$$

n = 3 gives

$$k_3(x,t) = \int_t^x k(x,z)k_2(z,t) = \int_t^x e^{3(x-z)}(z-t)e^{3(z-t)}dz$$

$$= = e^{3(x-t)}\int_t^x (z-t)dz = e^{3(x-t)}\frac{(x-t)^2}{2!}$$

Similarly we can write.

$$k_n(x,t) = e^{3(x-t)}\frac{(x-t)^{n-1}}{(n-1)!},\ n = 1,2,3.........$$

Resolvent kernel

R(x,t;λ)=

$$\sum_{m=1}^{\infty}\lambda^{m-1}k_m(x,t) = k_1(x,t)+\lambda k_2(x,t)+\lambda^2 k_3(x,t)+...$$

$$= e^{3(x-t)} + e^{3(x-t)}\frac{\lambda(x-t)}{1!} + e^{3(x-t)}\frac{[\lambda(x-t)]^2}{2!}+....$$

$$= e^{3(x-t)}\left[1+\frac{\lambda(x-t)}{1!}+\frac{[\lambda(x-t)]^2}{2!}+....\right]$$

$$= e^{3(x-t)}e^{\lambda(x-t)} = e^{3(x-t)+\lambda(x-t)} = e^{(3+\lambda)(x-t)}$$

**9. Which is not the Euler-lagrange equation for variational problems/निम्न में से कौन सी भिन्नतायी समस्याओं के लिए आयलर लेगरांजे समीकरण नहीं है-**

(a) $\frac{d}{dx}\left(\frac{\partial\rho}{\partial y'}\right)-\left(\frac{\partial\rho}{\partial y}\right)=0$

(b) $\frac{d^2}{dx^2}\left(\frac{\partial\rho}{\partial y}\right)-\frac{\partial\rho}{\partial y'}=0$

(c) $\frac{\partial\rho}{\partial x}-\frac{d}{dx}\left(\rho-y'\frac{\partial\rho}{\partial y'}\right)=0$

(d) $\frac{\partial\rho}{\partial y}-\frac{\partial^2\rho}{\partial x\partial y'}-y'\frac{\partial^2\rho}{\partial y\partial y'}-y''\frac{\partial^2\rho}{\partial y'^2}=0$

**Ans. (b) :** $\frac{d^2}{dx^2}\left(\frac{\partial\rho}{\partial y}\right)-\frac{\partial\rho}{\partial y'}=0$

Let J[y] be a functional of the form-

$$\int_a^b \rho(x,y,y')dx.$$

defined on the set of the functions y(x) which have continuous first derivatives in [a,b] and satisfy the boundary condition y (a) = A, y(b) = B.

Then a necessary condition for J[y] to have an extremum for a given function y(x) is that y(x) satisfy Euler-Lagrange equation

$$\frac{\partial\rho}{\partial y}-\frac{d}{dx}\left(\frac{\partial\rho}{\partial y'}\right)=0 \qquad ...(1)$$

Now $\frac{d\rho}{dx}=\frac{\partial\rho}{\partial x}.\frac{dx}{dx}+\frac{\partial\rho}{\partial y}.\frac{dy}{dx}+\frac{\partial\rho}{\partial y'}.\frac{dy'}{dx}$

$$\Rightarrow \frac{d\rho}{dx}=\frac{\partial\rho}{\partial x}+\frac{\partial\rho}{\partial y}.y'+\frac{\partial\rho}{\partial y'}.y'' \qquad ...(2)$$

and $\frac{d}{dx}\left(y'\frac{\partial\rho}{\partial y'}\right)=y'\frac{d}{dx}\left(\frac{\partial\rho}{\partial y'}\right)+\frac{\partial\rho}{\partial y'}y''$ ——(3)

from (i) and (ii) we obtain

$$\frac{d\rho}{dx}-\frac{d}{dx}\left(y'\frac{\partial\rho}{\partial y'}\right)=\frac{\partial\rho}{\partial x}+y'\frac{\partial\rho}{\partial y}-y'\frac{d}{dx}\left(\frac{\partial\rho}{\partial y'}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\rho-y'\frac{\partial\rho}{\partial y'}\right)-\frac{\partial\rho}{\partial x}=y'\left(\frac{\partial\rho}{\partial y}-\frac{d}{dx}\left(\frac{\partial\rho}{\partial y'}\right)\right)$$

$$\Rightarrow \frac{d}{dx}\left(\rho-y'\frac{\partial\rho}{\partial y'}\right)-\frac{\partial\rho}{\partial x}=0 \qquad \text{(from (1))}$$

Now, on expanding (1) we have-

$$\frac{\partial\rho}{\partial y}-\frac{\partial^2\rho}{\partial x\partial y'}\frac{dx}{dx}-\frac{\partial^2\rho}{\partial y\partial y'}\frac{dy}{dx}-\frac{\partial^2\rho}{\partial y'^2}\frac{dy'}{dx}=0$$

or $\frac{\partial\rho}{\partial y}-\frac{\partial^2\rho}{\partial x\partial y'}-y'\frac{\partial^2\rho}{\partial y\partial y'}-y''\frac{\partial^2\rho}{\partial y'^2}=0$

**10. A surface that is every where tangent to both flow velocity and vorticity is called/प्रवाह वेग एवं भ्रमिल दोनो की सर्वत्र स्पर्शी कहलाती है -**

(a) a steady tube/एक अपरिवर्ती नली

(b) an angular tube/एक कोणीय नली

(c) a vortex tube /एक भ्रमिल नली

(d) None of these /इनमें से कोई नहीं

**Ans. (c) :** a vortex tube.

A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector. The vortex lines drawn through each point of a closed curve constitute the surface of a vortex tube.

Hence, a surface that is every where tangent to both flow velocity and vorticity is called a vortex tube.

**11. Solution of the initial value problem u' = $-2tu^2$ u(0) = 1 with h = 0.2 on the interval [0, 1] is प्रारम्भिक मान समस्या u' = $-2tu^2$ u(0) = 1 जबकि अंतराल [0, 1] पर h = 0.2 का हल है**

(a) $u(0.2)= u_1= 0.8615241$

(b) $u(0.2)= u_1= 0.7615241$

(c) $u(0.2)= u_1= 0.9615241$

(d) $u(0.2)= u_1= 0.5615241$

**Ans. (c) :** $u(0.2) \approx u_1 = 0.9615328$.

The Runge-Kutta formula (fourth-order) for $f(t,u) = -2tu^2$ is given by

$$u(0.2) \approx u_1= u_0+\frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

where $k_1 = hf(t_0,u_0) = 0$

$$k_2 = hf\left(t_0+\frac{h}{2},u_0+\frac{k_1}{2}\right)=hf(0.1,1)=-0.04$$

$$k_3 = hf\left(t_0+\frac{h}{2},u_0+\frac{k_2}{2}\right)$$

$$= hf(0.1,0.98) = -0.038416$$

$$k_4 = hf(t_0+h,u_0+k_3)$$

$$= hf(0.2,0.961584) = -0.0739715$$

Thus,

$$u(0.2) \approx u_1 = 1+\frac{1}{6}\left[0-0.08-0.076832-0.0739715\right]$$
$$\approx 0.9615328.$$

**12. Solution of the difference equation $\Delta^2 y_j + 3\Delta y_j - 4y_j = j^2$ with initial conditions $y_0 = 0, y_2 = 2$, is**
**अंतर समीकरण $\Delta^2 y_j + 3\Delta y_j - 4y_j = j^2$ का हल प्रारम्भिक प्रतिबंध $y_0 = 0, y_2 = 2$ है**

(a) $y_j = \frac{1}{160}\left[63(-3)^j + 32(2)^j - 40j^2 - 60j - 95\right]$

(b) $y_j = \frac{63}{160}\left[32(2)^j - 40j^2 - +60j - 95\right]$

(c) $y_j = \frac{32}{160}[63(-3)^j - 40j^2 - 60j - 95]$

(d) $y_j = \frac{1}{160}\left[63(-3)^j - 32(2)^j - 40j^2 - 60j - 95\right]$

**Ans. (*) :** $(\Delta^2 + 3\Delta - 4)\, y_j = 0$
$\Delta^2 + 3\Delta - 4 = 0$ (Auxiliary oquation)
$[\because \Delta = E - 1]$
$(E-1)^2 + 3(E-1) - 4 = 0$
$E^2 - 2E + 1 + 3E - 3 - 4 = 0$
$E^2 + E + 1 - 3 \times 1 - 4 = 0$
$E^2 + E + 1 - 3 - 4$
$E^2 + E - 6 = 0$
$E^2 + (3-2)E - 6 = 0$
$E(E+3) - 2(E+3) = 0$
$(E+3)(E-2) = 0$
$E = 2, -3$
$y_j = C_1 2^j + C_2 (-3)^j$, Which is C.F. of given difference equation.
Given, $y_0 = 0, y_2 = 2$
$\Rightarrow \quad y_0 = C_1 2^0 + C_2(-3)^0$
$C_1 + C_2 = 0$
and $\quad y_2 = C_1 2^2 + C_2(-3)^2$
$2 = 4C_1 + 9C_2$
$\Rightarrow \quad 2 = -4C_2 + 9C_2$
$\Rightarrow \quad 5C_2 = 2$
$C_2 = 2/5$
$\therefore \quad C_1 = -2/5$

Now, P.I. = $\frac{j^2}{\phi(E)}$

Let, $j^2 = Aj^2 + Bj + C$ ............(ii)
$= Aj(j-1) + Bj + C$
$= A(j^2 - j) + Bj + C$
$= j^2(A) + j(B-A) + C$
$\therefore \quad A = 1$
$B - A = 0$
$B = 1$ and $C = 0$
$\therefore j^2 = j^2 + j$ [from (ii)]

P.I. = $\frac{j^2 + j}{\Delta^2 + 3\Delta - 4}$

$$\frac{j^2+j}{\Delta^2+3\Delta-4}$$
$$= \frac{j^2+j}{-4\left(1-\left(\frac{\Delta^2+3\Delta}{4}\right)\right)}$$
$$= -\frac{1}{4}\left(j^2+j\right)\left[1-\left(\frac{\Delta^2+3\Delta}{4}\right)\right]^{-1}$$
$$= -\frac{1}{4}\left[1+\frac{\Delta^2+3\Delta}{4}+\left(\frac{\Delta^2+3\Delta}{4}\right)^2+...........\right]\left(j^2+j\right)$$
$$= -\frac{1}{4}\left(1+\frac{3}{4}\Delta+\frac{\Delta^2}{4}+\frac{9\Delta^2}{16}\right)j^2+j$$
$$= -\frac{1}{4}\left(1+\frac{3}{4}\Delta+\frac{13\Delta^2}{16}\right)\left(j^2+j\right)$$
$$= -\frac{1}{4}\left[j^2+j+\frac{3}{4}(2j+1)+\frac{13}{16}(2)\right]$$
$$= -\frac{1}{4}\left(j^2+\frac{3}{2}j+j+\frac{3}{4}+\frac{13}{8}\right)$$
$$= -\frac{1}{4}\left(j^2+\frac{5}{2}j+\frac{19}{8}\right)$$
$$= -\frac{1}{32}\left(8j^2+20j+19\right)$$

So, the complete solution of the given difference equation = C.F. + P.I.

$$y_j = (-2/5)\,2^j + 2/5\,(-3)^j - \frac{1}{32}\left(8j^2+20j+19\right)$$
$$= -\frac{2}{5}2^j + \frac{2}{5}(-3)^j - \frac{\left(8j^2+20j+19\right)}{32}$$
$$= \frac{-64.2^j + 64(-3)^j - \left(8j^2+20j+19\right)5}{160}$$
$$= \frac{-64.2^j + 64(-3)^j - \left(40j^2+100j+95\right)}{160}$$

**13. Turbulence problem particularly depend on the term of the Navier Stokes equations which is the**
**प्रक्षोभ समस्या, नैवियर स्टेक्स समीकरण के जिस पद पर विशेषता रूप से आधारित होती है, वह है-**

(a) rate of change term /दर परिवर्तन पद
(b) convection term/संवहन पद
(c) source term/स्रोत पद
(d) diffusion term /विसरण पद

**Ans. (d) :** diffusion term
Fluid flow which is unsteady, irregular, seemingly random, and chaotic is called turbulent. The characteristic feature of turbulent flow is that the fluid velocity varies significantly and irregularly both in position and time. Turbulence, is the result of diffusion

in the flow. Diffusion is related to the stress tensor and to the viscosity of the fluid. Hence diffusion term of the Navier Stokes equations are particularly needed to describe turbulence problem.

**14. The Lagrangian for a charged particle in a electromagnetic field is-/इलेक्ट्रोमेग्निटिक क्षेत्र में आवेशित कण के लिए लेग्रेन्जियन है-**

(a) $L = T - q\phi + q(v.A)$
(b) $L = T + q\phi + q(v.A)$
(c) $L = T - q\phi - q(v.A)$
(d) $L = T + q\phi - q(v.A)$

**Ans. (a) :** $L = T - q\phi + q(v.A)$

A charged particle of charge q and mass m moving in an electric field $\vec{E}$ and a magnetic field $\vec{B}$ ,classically is subjected to the force $\vec{F}$ acting on the particle which is given by the Lorentz force law i.e.

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

where $\vec{v}$ is the instantaneous velocity of the particle.

If $\vec{A}$ is the vector potential and $\phi$ the scalar potential, then the magnetic field $\vec{B}$ and electric field $\vec{E}$ are written in terms of $\vec{A}$ and $\phi$ as

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

We assume that $\vec{A}$ and $\phi$ are function of the position vector $\vec{r}$ and time i.e. $\phi = \phi(\vec{r}, t)$ and $\vec{A} = \vec{A}(\vec{r}, t)$. The lagrangian of the system is written as

$$L = T - U$$

where the kinetic energy T and the velocity dependent potential energy U are given by

$$T = \frac{1}{2} m\vec{v}^2$$

$$U = q\phi - q\vec{v}.\vec{A}.$$

Thus the lagrangian L of a charged particle in an electromagnetic field is written as

$$L = \frac{1}{2}m\vec{v}^2 - q\phi + q\vec{v}.\vec{A}$$

The classical path of the charged particle is given by the principle of the least action with the lagrangian described above.

**15. The constraints on bead on a uniformly rotating wire in a force free space is**
**बल रहित समष्टि में एकसमान घूर्णीय तार बीड़ पर प्रतिबन्ध है -**

(a) Rheonomous /रिहनोमॉस
(b) Scleronomous /सदिशोनोमॉस
(c) No Constraints /कोई प्रतिबन्ध नहीं
(d) None of these /इनमें से कोई नहीं

**Ans. (a):** Rheonomous

A straight wire is pivoted at the origin and is arranged to swing around in horizontal plane at a constant angular speed ω. A bead of mass m slides frictionlessly along the straight wire. Generalized coordinates of the bead at radial position r is given by

$$x = r\cos\theta, \ y = r\sin\theta$$

with $\theta = \omega t$. Hence time dependent constraint. Thus, the system is Rheonomous.

**16. Except at origin where r = 0, the vortex flow is,**
**मूल बिंदु के अतिरिक्त, जहाँ r = 0 भ्रमिल प्रवाह है**

(a) rotational /घूर्णीय  (b) laminar/अप्रक्षुब्ध
(c) irrotational / अघूर्णीय  (d) turbulent/प्रक्षोभित

**Ans. (c) :** irrotational

Vortex flow is defined as the flow of a fluid along a curved path or the flow of a rotating mass of fluid is known as vortex flow.

When no external torque is required to rotate the fluid mass the flow is called free vortex flow. Hence in a free vortex flow total mechanical energy remains constant. there is neither any energy interaction between an outside source and the flow. The fluid rotates by virtue of some rotation previously imparted to it or because of some internal action.

Now, constancy of total mechanical energy in the entire flow field implies the irrotationality of the flow.

Hence, free vortex flow is irrotational except at r = 0 which in practice is impossible.

Vortex flow is irrotational every where except at the point r = 0, where the velocity is infinite. Therefore, the origon, r = 0 is a singular point in the flow field.

**17. Using Runge-Kutta method of order 4 for the following initial value problem $\frac{dy}{dx} = x^2 + y^2, y(1) = 0$ the value of y(1.1) shall be**
**चार कोटि में रंगा-कुत्ता विधि द्वारा निम्न प्रारम्भिक मान समस्या $\frac{dy}{dx} = x^2 + y^2, y(1) = 0$ के लिए y(1.1) का मान होगा**

(a) 0.110  (b) 0.117
(c) 0.119  (d) 0.101

**Ans. (b) :** 0.117

Here $\frac{dy}{dx} = x^2 + y^2$, y(1) = 0, h = 0.1 and $f(x_0, y_0) = 1$

Fourth- order Runge-Kutta formula is given by

$$y(1.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0 y_0) = 0.1(1) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f(1.05, 0.05)$$

$$= 0.1105$$

$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$ =0.1f(1.05,0.05525)

= 0.1105

$k_4 = hf(x_0+h, y_0+k_3)$ = 0.1f(1.1, 0.1105)

= 0.1222

Thus

$y(1.1) = 0 + \frac{1}{6}(0.1+2(0.1105) + 2(0.1105) + 0.1222)$

= 0.1116

≈ 0.117

**18. Using Euler's modified method. the value of y(0.02) for the differential equation $\frac{dy}{dx} = x^2 + y, y(0) = 1$ and taking h = 0.01 is.**

**ऑइलर संशोधित विधि के प्रयोग से y(0.02) का मान निम्न अवकल समीकरण के लिए तथा h = 0.01 मानते हुए $\frac{dy}{dx} = x^2 + y, y(0) = 1$ होगा**

(a) y(0.02) = 1.2020 (b) y(0.02) = 1.0250

(c) y(0.02) = 1.0201 (d) y(0.02) = 1.0203

**Ans. (d) :** y(0.02) = 1.0203

Here $\frac{dy}{dx} = x^2 + y, y(0) = 1$ and taking h = 0.01, $f(x_0y_0)$ =1

By Euler's formula we have

$y(0.01)^{(0)} = y(0)+hf(x_0y_0) = 1.01$

Further $x_1$ = 0.01 and $f(x_1, y(0.01)^{(0)}) = f(0.01,1.01) = 1.0101$

Now by iteration formula we have

$y(0.01)^{(1)} = y(0) + \frac{h}{2}\left[f(x_0,y_0)+f(x_1,y(0.01)^{(0)})\right]$

= 1.0100

Again,

$y(0.01)^{(2)} = y(0) + \frac{h}{2}\left[f(x_0,y_0)+f(x_1,y(0.01)^{(1)})\right]$

= 1.0100

Next with $x_1$= 0.01, $y_1$ =1.0100 and h = 0.01 we continue the procedure to obtain y(0.02) i.e. the value of y when x = 0.02.

By Euler's formula,

$y(0.02)^{(0)} = y(0.01)+hf(x_1,y_1) = 1.0201$

∴ $f(x_1,y(0.02)^{(0)}) = f(0.01,1.0201) = 1.0202$

By iteration formula we have

$y(0.02)^{(1)} = y(0.01) + \frac{h}{2}\left[f(x_1,y_1)+f(x_1,y(0.02)^{(0)})\right]$

≈ 1.0201

Again,

$Y(0.02)^{(2)} = y(0.01) + \frac{h}{2}\left[f(x_1,y_1)+f(x_1,y(0.02)^{(1)})\right]$

≈ 1.0203.

**19. The number of basic variables in a transportation problem are at the most**

**परिवहन समस्या में मूल चरों की अधिकतम संख्या है**

(a) m + n –1 (b) m – n +1

(c) m + n +1 (d) m – n –1

**Ans. (a) :** m + n –1

The number of basic variables in m×n balanced transportation problem is at most m + n –1.

**20. Solution of the following game problem.**

**निम्न खेल समस्या का हल है**

Player B

Player A $\begin{bmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{bmatrix}$

(a) $\left(\frac{5}{14},\frac{9}{14}\right)\left(\frac{9}{14},0,\frac{5}{14}\right)v = \frac{73}{14}$

(b) $\left(\frac{9}{14},0,\frac{5}{14}\right)\left(\frac{5}{14},\frac{9}{14}\right)v = \frac{73}{14}$

(c) $\left(\frac{9}{14},\frac{7}{14},\frac{5}{14}\right)\left(\frac{9}{14},\frac{5}{14}\right)v = \frac{73}{14}$

(d) None of these /इनमें से कोई नहीं

**Ans. (b) :** $\left(\frac{9}{14},0,\frac{5}{14}\right)\left(\frac{5}{14},\frac{9}{14}\right)v = \frac{73}{14}$

| Player A \ Player B | $B_1$ | $B_2$ | Row Minimum |
|---|---|---|---|
| $A_1$ | 2 | 7 | 2 |
| $A_2$ | 3 | 5 | 3 |
| $A_3$ | 11 | 2 | 2 |
| Column Maximum | 11 | 7 | |

maximin = 3 and minimax = 7 and clearly no saddle point. The value of the game v lies between 3 and 7 i.e 3 < v < 7.

Let $p_1$, $p_2$, $p_3$ and $q_1$, $q_2$ be the probabilities of selecting strategies $A_1$, $A_2$, $A_3$ and $B_1$, $B_2$ by player A and player B respectively.

Then,

$2q_1 + 7q_2 \le v$

$3q_1 + 5q_2 \le v$

$11q_1 + 2q_2 \le v$

and $q_1 + q_2 = 1$

$q_1, q_2 \ge 0$

$\Rightarrow \frac{2q_1}{v} + \frac{7q_2}{v} \le 1$

$\frac{3q_1}{v} + \frac{5q_2}{v} \le 1$

$\frac{11q_1}{v} + \frac{2q_2}{v} \le 1$

and $\frac{q_1}{v} + \frac{q_2}{v} = \frac{1}{v}$

Let $\frac{q_1}{v} = x_1$ and $\frac{q_2}{v} = x_2$ ...(*)

Therefore the problem is to maximize $\frac{1}{v} = x_1 + x_2$

subject to

$2x_1 + 7x_2 \leq 1$

$3x_1 + 5x_2 \leq 1$

$11x_1 + 2x_2 \leq 1$

$x_1, x_2 \geq 0$

Now using Simplex method

After introducing slack variables

Max $Z = x_1 + x_2 + 0S_1 + 0S_2 + 0S_3$

subject to

$2x_1 + 7x_2 + S_1 = 1$

$3x_1 + 5x_2 + S_2 = 1$

$11x_1 + 2x_2 + S_3 = 1$

and $x_1, x_2, S_1, S_2, S_3 \geq 0$

| | | $C_j$ | 1 | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| B | $C_B$ | $X_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | Min Ratio $\frac{X_B}{x_1}$ |
| $S_1$ | 0 | 1 | 2 | 7 | 1 | 0 | 0 | $\frac{1}{2} =$ 0.5 |
| $S_2$ | 0 | 1 | 3 | 5 | 0 | 1 | 0 | $\frac{1}{3} =$ 0.3333 |
| $S_3$ | 0 | 1 | 11 | 2 | 0 | 0 | 1 | $\frac{1}{11} =$ 0.0909→ |
| Z = 0 | | $Z_j$ | 0 | 0 | 0 | 0 | 0 | |
| | | $Z_j - C_j$ | −1 ↑ | −1 | 0 | 0 | 0 | |

Negative minimum $Z_j - C_j$ is −1 in column 1 and minimum ratio is 0.0909 in row 3, so entering variable is $x_1$ & leaving variable is $S_3$.

∴ The pivot element is 11.

| | | $C_j$ | 1 | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| B | $C_B$ | $X_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | Min ratio |
| $S_1$ | 0 | $\frac{9}{11}$ | 0 | $\frac{73}{11}$ | 1 | 0 | $-\frac{2}{11}$ | $\frac{\frac{9}{11}}{\frac{73}{11}} = \frac{9}{73}$ = 0.1233 |
| $S_2$ | 0 | $\frac{8}{11}$ | 0 | $\frac{49}{11}$ | 0 | 1 | $-\frac{3}{11}$ | $\frac{\frac{8}{11}}{\frac{49}{11}} = \frac{8}{49}$ 0.1633 |
| $x_1$ | 1 | $\frac{1}{11}$ | 1 | $\frac{2}{11}$ | 0 | 0 | $\frac{1}{11}$ | $\frac{\frac{1}{11}}{\frac{2}{11}} = \frac{1}{11}$ 0.5 |
| Z = $\frac{1}{11}$ | | $Z_j$ | 1 | $\frac{2}{11}$ | 0 | 0 | $\frac{1}{11}$ | |
| | | $Z_j$-$C_j$ | 0 | $-\frac{9}{11}$ | 0 | 0 | $\frac{1}{11}$ | |

Negative minimum $Z_j - C_j$ is $-\frac{9}{11}$ in column 2 and minimum ratio is 0.1233 in row 1, so entering variable is $x_2$ & leaving variable is $S_1$.

∴ The pivot element is $\frac{73}{11}$.

| Iteration-3 | | $C_j$ | 1 | 1 | 0 | 0 | 0 | |
|---|---|---|---|---|---|---|---|---|
| B | $C_B$ | $X_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | Min ratio |
| $x_2$ | 1 | $\frac{9}{73}$ | 0 | 1 | $\frac{11}{73}$ | 0 | $-\frac{2}{73}$ | |
| $S_2$ | 0 | $\frac{13}{73}$ | 0 | 0 | $-\frac{49}{73}$ | 1 | $-\frac{11}{73}$ | |
| $x_1$ | 1 | $\frac{5}{73}$ | 1 | 0 | $-\frac{2}{73}$ | 0 | $\frac{7}{73}$ | |
| Z = $\frac{14}{73}$ | | $Z_j$ | 1 | 1 | $\frac{9}{73}$ | 0 | $\frac{5}{73}$ | |
| | | $Z_j$-$C_j$ | 0 | 0 | $\frac{9}{73}$ | 0 | $\frac{5}{73}$ | |

Since all $Z_j - C_j \geq 0$

The, optional solution is arrived with value of variables as :

$x_1 = \frac{5}{73}, x_2 = \frac{9}{73}$ &

Max $Z = \frac{14}{73}$

Hence, back substituting from (*) gives optimal strategies for player B = $\left(\frac{5}{14}, \frac{9}{14}\right)$ and optimal strategies for player A = $\left(\frac{9}{14}, 0, \frac{5}{14}\right)$.

Hence the solution of the given game problem is

$\left(\frac{9}{14}, 0, \frac{5}{14}\right)\left(\frac{5}{14}, \frac{9}{14}\right) v = \frac{73}{14}$

**21. If either the primal or the dual problem has a finite optimal solution then the other problem also has/यदि किसी अद्वैत अथवा द्वैत समस्या के सीमित इष्टतम हल हो तब दूसरी समस्या के भी-**

(a) a finite optimal solution./सीमित इष्टतम हल होगा

(b) an infinite optimal solution./असीमित इष्टतम हल होगा

(c) no optimal solution. /कोई इष्टतम हल नहीं होगा

(d) None of these/इनमें से काई नहीं

**Ans. (a) :** a finite optimal solution.

For every linear programming problem, there is a corresponding unique linear programming problem called the dual of the original problem called primal problem.

If the optimal solution of either problem (primal or dual) is known then the optimal solution of the other is also available.

**22. If corresponding to any negative $\Delta_j = (z_j - c_j)$ all elements of the column $X_j$ are negative or zero (≤0), then the solution under test will be**

**यदि किसी ऋणात्मक $\Delta_j = (z_j - c_j)$ के सापेक्ष, कॉलम $X_j$ के सभी अवयव ऋणात्मक अथवा शून्य (≤0) हो तब परीक्षण के अन्तर्गत हल होगा-**

(a) Bounded /परिबद्ध

(b) Unbounded/अबाध

(c) Suboptimal/उपइष्टतम

(d) No solution/कोई हल नहीं

**Ans. (b) :** Unbounded.

Under the simplex method the leaving variable is determined by using a ratio test for every constraint row i, compute the ratio $\frac{b_i}{a_{is}}$, if $a_{is} > 0$ where column s is the pivot column. That is ,we divide the right hand side of each constraint by the element in the pivot column of the same row, but only if the denominator $a_{is}$ is strictly positive in value.

Now if for any tableau $c_j - z_j$ indicates that a nonbasic variable should enter the basic, but no ratios can be computed for the constraints because every constraint coefficient in the pivot column is either zero, or negative, the problem is unbounded. Increasing the value of the entering variable improves the objective function without limit.

**23. For an LPP,**

**Maximum $Z = 6x_1 - 4x_2$**

**Subject to : $x_1+x_2 \le 2$**

**$x_1+x_2 \ge 4$**

**$x_1, x_2 \ge 0$**

**If has -**

**LPP के लिए**

**Maximum $Z = 6x_1 - 4x_2$**

**Subject to : $x_1+x_2 \le 2$**

**$x_1+x_2 \ge 4$**

**$x_1, x_2 \ge 0$**

**दर्शाता है –**

(a) No solution due to inconsistent system of constraints. /असंगत प्रतिबन्ध व्यवस्था के कारण कोई हल नहीं।

(b) No solution due to consistent system of constraints./संगत प्रतिबन्ध व्यवस्था के कारण हल नहीं।

(c) Infinite many solution./अनन्त अनेक हल ।

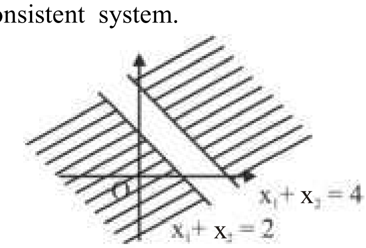
(d) LPP is unbounded./ LPP अबाध है ।

**Ans. (a) :** No solution due to inconsistent system of constraints.

Given system of constraints are inconsistent because

$x_1 + x_2 \le 2$ and $x_1 + x_2 \ge 4$

can not happen simultaneously i.e. there is no feasible region between them, so there is no solution to the given inconsistent system.




**24. If the prime p ≥ 7, then how many consecutive quadratic residues (mod p ) will be**

**यदि अभाज्य p ≥ 7 तब (mod p ) में कितने क्रमिक द्विघातीय अवशेष होगें ?**

(a) Two/दो (b) Three/तीन

(c) Four/चार

(d) None of these/इनमें से कोई नहीं

**Ans. (a) :** Two

**Definition-** The Legendre Symbol

$$\left(\frac{a}{p}\right)$$

is assigned the value of 1 if a is a quadratic residue of p. Otherwise, it is assigned the value of –1.

**Proposition-** For any prime p > 5 there exists integer $1 \le a \le p-1$ for which $\left(\frac{a}{p}\right) = \left(\frac{a+1}{p}\right) = 1$.

That is there are consecutive quadratic residues of p.

Proof - Since $x^2 \equiv 1$, $x^2 \equiv 4$ and $x^2 \equiv 9$ have solution for all p > 5 then consider $x^2 \equiv 2$, $x^2 \equiv 5$, $x^2 \equiv 10$.
Now for p > 5; gcd(2,p) = 1, gcd(5,p) = 1 so gcd(10,p) = 1 and because if p is an odd prime and gcd (ab, p) = 1 then at least one of a, b or ab is a quadratic residue of p. We have at least one of 2, 5 or 10 must be a quadratic residue of p.

If $\left(\frac{2}{p}\right) = 1$, then 1 and 2 are consecutive residues.

If $\left(\frac{5}{p}\right) = 1$, then 4 and 5 are consecutive residues.

If $\left(\frac{10}{p}\right) = 1$, then 9 and 10 are consecutive residues.

Thus , the above showed that at least one pair of consecutive residues for p > 5.

**25. If p is an odd prime and if g is a primitive root (mod p ), then $\left(\frac{g}{r}\right) =$**
**यदि p एक विषम अभाज्य तथा यदि g एक पूर्वग मूल (mod p )हो तब $\left(\frac{g}{r}\right) =$**

(a) –1 (b) 1
(c) ± 1
(d) None of these/इनमें से कोई नहीं

**Ans. (a) :** –1
If g is a primitive root modulo p then order of g modulo p is $\phi(p) = p - 1$ i.e.

$$g^{p-1} \equiv 1 \bmod p$$

$$\Rightarrow \quad p \mid g^{p-1} - 1$$

$$\Rightarrow \quad p \mid \left(g^{\frac{p-1}{2}} - 1\right)\left(g^{\frac{p-1}{2}} + 1\right)$$

Now if $p \mid g^{\frac{p-1}{2}} - 1$ then we have $g^{\frac{p-1}{2}} \equiv 1 \pmod p$ which contradicts that g is primitive root modulo p.
Hence we must have,

$$p \mid g^{\frac{p-1}{2}} + 1 \qquad \Rightarrow g^{\frac{p-1}{2}} \equiv -1 \pmod p.$$

For an odd prime p and an integer g relative prime to p we have $\left(\frac{g}{p}\right) \equiv g^{\frac{p-1}{2}} \equiv -1 \pmod p$

where $\left(\frac{g}{p}\right)$ denotes the Legendre symbol.

**Note :** An educated guess can be made that the examiner had intended to ask the $\left(\frac{g}{p}\right) = -1$

**26. If p is prime , then $(a + b)^p \equiv$**
**यदि अभाज्य है, तब $(a + b)^p \equiv$**

(a) $a^p + b^p \pmod p$ (b) $a^p + b^p \pmod N$
(c) $a^p - b^p \pmod p$ (d) $a^p - b^p \pmod N$

**Ans. (a) :** $a^p + b^p \pmod p$
Now if p is a prime , then
${}^pC_i \equiv 0 \pmod p$ for $1 \le i \le p-1$
Thus by binomial theorem we have

$$(a+b)^p \equiv {}^pC_0 a^p + {}^pC_1 a^{p-1} b + \dots + {}^pC_{p-1} a b^{p-1} + {}^pC_p b^p$$
$$\equiv a^p + b^p \pmod p$$

**27. Let a and b belong to set S, Let R be an equivalence relation on S. Then ${}_aR_b$ if and only if**
**a एंव b समुच्चय S में है। R का S से तुल्यता संबंध है। तब ${}_aR_b$ यदि केवल यदि ,**

(a) $[a]_R \ne [b]_R$ (b) $[a]_R \subseteq [b]_R$
(c) $[a]_R \subset [b]_R$ (d) $[a]_R = [b]_R$

**Ans. (d) :** $[a]_R = [b]_R$
**Proposition-** An equivalence relation on a set S determines a partition of S.
Given an equivalence relation, one defines a partition this way : The subset that contains a is the set of all elements b such that ${}_aR_b$. This subset is called the equivalence class of a.
$\Rightarrow [a]_R = \{b \in S|_a R_b\}$.
**Claim-** The subset of S that are equivalence classes of partition S.
**Proof-** The reflexive axiom tells us that a is in its equivalence class. Therefore the class $[a]_R$ is non empty, and since a can be any element, the union of the equivalence classes is the whole set S.
If $[a]_R$ and $[b]_R$ have an element in common, say d. If x is in $[b]_R$ then ${}_bR_x$. Since d is in both sets, ${}_aR_d$ and ${}_bR_d$, and the symmetry property tells us that ${}_dR_b$. So we have ${}_aR_d$, ${}_dR_b$, and ${}_bR_x$. Two applications of transitivity show that ${}_aR_x$. and therefore that x is in $[a]_R$ showing that
$[b]_R \subset [a]_R$.
Similarly ; it can be shown that $[a]_R \subset [b]_R$.
Hence, we get that $[a]_R = [b]_R$.

**28. A positive integer which has more divisors than any smaller positive integer is called**
**एक धनात्मक पूर्ण संख्या जिसके विभाजकों की संख्या किसी अन्य छोटी धनात्मक पूर्ण संख्या की अपेक्षा अधिक होते हैं, कहलाती है-**

(a) Highly Composite Number/उच्च भाज्य संख्या
(b) Ramanujan Composite Number/रामानुजन भाज्य संख्या
(c) Harshad Composite Number /हर्षद भाज्य संख्या
(d) Hardy Composite Number/हर्डी भाज्य संख्या

**Ans. (a) :** Highly Composite Number.
An integer n > 1 is termed highly composite if it has more divisors than any preceding integer ; in other words, the divisor function τ satisfies $\tau(m) < \tau(n)$ for all m < n. The first 10 highly composite numbers are 2,4,6,12,24,36,48,60,120 and 180.

**29. If a positive integer on a given base is divisible by the sum of its digits on the same base called as**
**दिए गए आधार पर धनात्मक पूर्ण संख्या यदि अंको के योग से उसी आधार पर भाज्य हो तो संख्या कहलाती है-**

(a) A Ramanujan Number /रामानुजन संख्या
(b) A Harshad Number/हर्षद संख्या
(c) A Hardy Number /हार्डी संख्या
(d) None of these /इनमें से कोई नहीं

**Ans. (b) :** A Harshad Number
A base b Harshad number (or Niven numbers) is a positive integer that is divisible by the sum of its base b digits. For example, in the decimal number system, 1729 is a Harshad number since 1+7+2+9 = 19, and 1729 = 19×91.

**30. If $2^n - 15 = x^2$ then**
**यदि $2^n - 15 = x^2$ तब**

(a) n = 2 or n = 4/ n = 2 अथवा n = 4
(b) n = 4 or n = 6/ n = 4 अथवा n = 6
(c) n = 2 or n = 8/ n = 2 अथवा n = 8
(d) n = 4 or n = 8/ n = 4 अथवा n = 8

**Ans. (b) :** n = 4 or n = 6
If $2^n - 15 = x^2$ ; x is an integer then either n = 4 or n = 6 because if n = 4 ; $2^n - 15 = 1$ and if n = 6 ; $2^n - 15 = 49$.

**31. Suppose that H, K are cyclic groups of order m, n respectively. Then H×K will be cyclic if**
**मान लीजिए कि H, K क्रमशः m, n कोटि के चक्रीय समूह है । तब H×K चक्रीय होगा, यदि**

(a) m, n are even integers./ m, n सम पूर्णांक हो ।
(b) m, n are odd integers / m, n विषम पूर्णांक हो
(c) m, n are odd prime numbers / m, n विषम अभाज्य संख्याएं हैं।
(d) m, n are relatively prime numbers. / m, n आपेक्षिक अभाज्य संख्याएं है

**Ans. (d) :** m, n are relatively prime numbers
**Proposition :** Let H and K be finite cyclic groups of order m and n respectively. Then H×K is cyclic if and only if m and n are relatively prime.
**Proof-** Assume H×K is cyclic. Because $|H| = m$ and $|K| = n$, so $|H \times K| = mn$.
Suppose gcd(m,n) = d and (h, k) is a generator of H×K.
Since $(h, k)^{mn/d} = ((h^m)^{n/d}, (k^n)^{m/d}) = (e_H, e_K)$, we have

$$mn = |(h,k)| \le \frac{mn}{d}$$

Thus, d = 1i.e. m & n relatively prime numbers.

**32. If G is an abelian group and the action of G on itself by conjugation is the trivial action g.a = a for all g, a∈G then for each a∈G the conjugacy class of a is,**
**यदि G एक आबेली समूह हो और संयुग्मी द्वारा G का स्वयं पर क्रिया सभी g, a∈G के लिए तुच्छ क्रिया g.a = a हो, तब प्रत्येक a∈G के लिए a का संयुग्मता वर्ग होगा -**

(a) {a}
(b) {G}
(c) a{G}
(d) None of these / इनमें से कोई नहीं

**Ans. (a) :** {a}
For an element a of a group G its conjugacy class is the set of elements conjugate to it

$$\{gag^{-1} : g \in G\}$$

If G is an abelian then every element is in its own conjugacy class :

$$g.a = gag^{-1} = a \text{ for all } g \in G$$

**33. Suppose there exist 5 groups of order $p^2q = 12$ then**
**मान लीजिए कि ( $p^2q = 12$ ) कोटि के 5 समूह अस्तित्व में है तो,**

(a) One of which is non-abelian. /उनमें से एक अन-आबेली है।
(b) Two of which are non-abelian./ उनमें से दो अन-आबेली हैं।
(c) Three of which are non-abelian./ उनमें से तीन अन-आबेली हैं।
(d) All five are non-abelian. / सभी पाँचों अन-आबेली हैं।

**Ans. (c) :** Three of which are non-abelian.
Structure Theorem for finite abelian groups dictates that any abelian group of order 12 can be written as a product of cyclic groups.
So, any abelian group of order 12 can be written as
$\mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_4 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,$
Clearly $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_6$
and $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$
Hence, there are only two isomorphism classes a groups of order 12 which are abelian.
**Note :** The examiner should have specified that groups of order 12 upto isomorphism class.

**34. Let the sets of rationals and reals numbers be denoted by Q and R respectively.**
**Choose the correct answer .**
**मान लिजिए कि परिमेय और वास्तविक संख्याओं के समुच्चयों को क्रमशः Q तथा R द्वारा दर्शाया गया है।**
**सही उत्तर को चुनिए -**

(a) Q is both a subring and an ideal of R / Q होगा R का उपवलय तथा गुणजावली दोनों।
(b) Q is a subring but not an ideal of R / Q होगा R का उपवलय परन्तु गुणजावली नहीं।
(c) Q is an ideal but not a subring of R / Q होगी R की गुणजावली परन्तु उपवलय नहीं।
(d) Q is neither a subring nor an ideal of R / Q न तो R का उपवलय और न ही गुणजावली है।

**Ans. (b):** Q is a subring but not an ideal of R
The set of rational numbers under the usual addition and multiplication of real numbers is a commutative ring with unit element. Hence, Q is a subring of R. Q is not an ideal of R because for any rational number r and real number $\sqrt{2}$ neither r$\sqrt{2}$ nor $\sqrt{2}$r belongs to Q.

**35. Let F be a field of integers modulo 11, then the polynomial $x^2 + x + 4$**
**माना कि मापांक 11 के पूर्णांकों का क्षेत्र F है, तब बहुपद $x^2 + x + 4$**

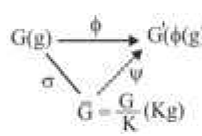
(a) is reducible over F/ F पर खंडनीय है।
(b) is irreducible over F/ F पर अखंडनीय है।
(c) has prime factor over F/ F पर अभाज्य गुणन खंड रखता है।
(d) None of these /इनमें से कोई नहीं

**Ans. (b) :** is irreducible over F
A polynomial of degree 2 or 3 over a field F is reducible if and only if it has a root in F.
Now field of integers modulo 11 = $Z_{11}$ ={0,1,2,3,4,5,6,7,8,9,10}
and $p(x) = x^2 + x + 4$ (mod 11)
Observe that
$P(0) = 0 + 0 + 4 \pmod{11} = 4$
$P(1) = 1 + 1 + 4 \pmod{11} = 6$
$P(2) = 4 + 2 + 4 \pmod{11} = 10$
$P(3) = 9 + 3 + 4 \pmod{11} = 5$
$P(4) = 16 + 4 + 4 \pmod{11} = 2$
$P(5) = 25 + 5 + 4 \pmod{11} = 1$
$P(6) = 36 + 6 + 4 \pmod{11} = 2$
$P(7) = 49 + 7 + 4 \pmod{11} = 5$
$P(8) = 64 + 8 + 4 \pmod{11} = 10$
$P(9) = 81 + 9 + 4 \pmod{11} = 6$
$P(10) = 100 + 10 + 4 \pmod{11} = 4$
Hence the polynomial $x^2 + x + 4$ is irreducible in $\square_{11}$ since it has no root in $\square_{11}$.

**36. Every Homomorphic of a group is isomorphic to**
**समूह के प्रत्येक समाहारी, तुल्यकारी होते है -**

(a) cyclic group /चक्रीय समूह के
(b) quotient group /विभाग समूह के
(c) normal subgroup /प्रसामान्य उपसमूह के
(d) none of these/इनमें से कोई नही

**Ans. (b) :** quotient group
**Theorem-** Let $\phi: G \to G'$ be a surjective group homomorphism with kernal K.
The quotient group $\bar{G} = \frac{G}{K}$ is isomorphic to the image G'.
To be precise, let $\sigma : G \to \bar{G}$ be the canonical map. There is a unique isomorphism $\Psi : \bar{G} \to G'$ such that
$\phi = \Psi o \sigma$




where $\sigma(g) = Kg$.

**37. If G is a finite group and $O(G) = p^n$, where p is prime number and n is positive integers, then**
**यदि G एक परिमित समूह है तथा $O(G) = p^n$ जहाँ p अभाज्य संख्या है तथा n धनात्मक पूर्णांक है तब,**

(a) $Z(G) \neq \{e\}$
(b) $Z(G) = \{e\}$
(c) $Z(G) = G$
(d) None of these / इनमें से कोई नही

**Ans. (a) :** $Z(G) \neq \{e\}$
Groups whose orders are positive powers of a prime p are called p-groups.
**Theorem-** The center of a p-group is not the trivial group.

**38. If V(F) and W(F) are finite dimensional vector spaces of dimensions n and m respectively. Then the space L(V,W) is finite dimensional of the dimension**
**यदि V(F) और W(F) क्रमशः n और m विमा के परिमित विमीय सदिश समष्टि है। तब समष्टि L(V,W) किस विमा की परिमित विमीय है?**

(a) m + n
(b) m × n
(c) m – n
(d) $\frac{m}{n}$

**Ans. (b) :** m × n
Let V and W be vector spaces over the field F. Let S and T be linear transformations from V into W. The function (S + T) defined by

$$(S + T)(v) = S(v) + T(v) \text{ for all } v \in V(F)$$

is a linear transformation from V into W. If c is any element of F. The function cS defined by

$$(cS)(v) = cS(v) \text{ for all } v \in V(F)$$

is a linear transformation from V into W. The set of all linear transformation from V into W, together with addition and scalar multiplication defined above, is a vector space over the field F denoted by L(V,W)
**Theorem-** If V and W are of dimensions m and n , respectively, over F, then the space L(V,W) is of dimension mn over F.

**39. The system of linear equations**
**$x – y + z = 2$**
**$x + y – z = 0$**
**$6x – 4y + 4z = 11$**
**रैखिक समीकरणों का तत्रं**
**$x – y + z = 2$**
**$x + y – z = 0$**
**$6x – 4y + 4z = 11$**

(a) has trivial solution /तुच्छ हल रखता है।
(b) has unique solution /अद्वितीय हल रखता है।
(c) is consistent /संगत है।
(d) is inconsistent /असंगत है।

**Ans. (d) :** is inconsistent

The system of linear equations

$$AX = B \equiv \begin{bmatrix}1 & -1 & 1\\1 & 1 & -1\\6 & -4 & 4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\0\\11\end{bmatrix}$$

Augmented matrix $A' = \begin{bmatrix}1 & -1 & 1 & 2\\1 & 1 & -1 & 0\\6 & -4 & 4 & 11\end{bmatrix}$ is to be row-reduced.

Now

$$\begin{bmatrix}1 & -1 & 1 & 2\\1 & 1 & -1 & 0\\6 & -4 & 4 & 11\end{bmatrix} \xrightarrow[R_3\to R_3-6R_1]{R_2\to R_2-R_1} \begin{bmatrix}1 & -1 & 1 & 2\\0 & 2 & -2 & -2\\0 & 2 & -2 & -1\end{bmatrix}$$

$$\xrightarrow{R_3\to R_3-R_2} \begin{bmatrix}1 & -1 & 1 & 2\\0 & 2 & -2 & -2\\0 & 0 & 0 & 1\end{bmatrix}$$

We observe that Rank(A) = 2 and Rank(A:b) = 3

$\because$ Rank(A) $\neq$ Rank (A:b) and hence given system is inconsistent.

**40. If $C([a,b])$ is the space of continuous real valued function defined on the closed interval [a,b] of real line then**

$L(x) = \int_a^b f(x)dx$ **defines the.**

**यदि वास्तविक रेखा के संवृत अंतराल [a,b] पर परिभाषित वास्तविक मान सतत फलन का समष्टि $C([a,b])$ हो तो**

$L(x) = \int_a^b f(x)dx$ **परिभाषित करता है-**

(a) function L on $C([a,b])$ / $C([a,b])$ पर फलन L

(b) linear function L on $C([a,b])$ / $C([a,b])$ पर रैखिक फलन L

(c) linear functional L on $C([a,b])$ / $C([a,b])$ पर रैखिक फलनक L

(d) transformation L on $C([a,b])$ / $C([a,b])$ पर रूपांतरण L

**Ans. (c) :** linear functional L on $C([a,b])$

Let [a, b] be a closed interval on the real line and let C([a,b]) be the space of continuous real-valued functions on [a,b].Then

$$L(x) = \int_a^b f(x)dx$$

defines a linear functional L on C([a, b]) because

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx \text{ and}$$

$$\int_a^b \alpha f(x)dx = \alpha\int_a^b f(x)dx$$

for every f, g$\in$C([a, b]) and for every $\alpha \in$ Field

**41. The matrix $\begin{bmatrix}a-ic & b-id\\-(b+id) & a+ic\end{bmatrix}$ will be unitary if and only if**

**अव्यूह $\begin{bmatrix}a-ic & b-id\\-(b+id) & a+ic\end{bmatrix}$ ऐकिक होगी यदि केवल यदि,**

(a) $a^2 - b^2 + c^2 - d^2 = 1$ (b) $-a^2 + b^2 - c^2 + d^2 = 1$

(c) $a^2 + b^2 + c^2 + d^2 = 1$ (d) $a^2 + b^2 + c^2 + d^2 \neq 1$

**Ans. (c) :** $a^2 + b^2 + c^2 + d^2 = 1$

If $\begin{bmatrix}a-ic & b-id\\-(b+id) & a+ic\end{bmatrix}$ is unitary then

$$\det\left(\begin{bmatrix}a-ic & b-id\\-(b+id) & a+ic\end{bmatrix}\right) = 1$$

$\therefore$ (a – ic)(a + ic) + (b + id)(b – id) = 1

$\Rightarrow a^2 + c^2 + b^2 + d^2 = 1$

**42. Orthonormal set of vectors in an inner product space is/आंतर गुणन समष्टि में सदिशों का प्रसामान्य लांबिक समुच्चय होता है**

(a) Linearly dependent /रैखिकतः आश्रित

(b) Linearly independent/ रैखिकतः स्वतंत्र

(c) Neither dependent nor independent/न तो आश्रित न ही स्वतंत्र

(d) None of these /इनमें से कोई नही

**Ans. (b) :** Linearly independent

**Proposition-** If $\{v_i\}$is an orthonomal set in an inner product space, then the vectors in $\{v_i\}$ are linearly independent.

**Proof-** Suppose that $\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n = 0$ therefore $0 = (\alpha_1v_1+\alpha_2v_2+...+\alpha_nv_n , v_i) = \alpha_1(v_1,v_i) +...+\alpha_n(v_n,v_i)$.

Since$(v_i,v_i) = \delta_{ij}$ this equation reduces to $\alpha_i = 0$ . Thus the $v_i$'s are linearly independent.

**43. If $\{\alpha,\beta\}$ is an orthonormal set in an inner product space then distance between $\alpha$ and $\beta$ is आंतर गुणन समष्टि में प्रसामान्य लांबिक समुच्चय $\{\alpha,\beta\}$ हो तो $\alpha$ एवं $\beta$ के मध्य दूरी है**

(a) 0 (b) 1

(c) 2 (d) $\sqrt{2}$

**Ans. (d) :** $\sqrt{2}$

Distance between $\alpha$ and $\beta$ = $d(\alpha,\beta) = \sqrt{\|\alpha-\beta\|^2}$

$$= \sqrt{(\alpha-\beta,\alpha-\beta)}$$

$$= \sqrt{(\alpha,\alpha-\beta)-(\beta,\alpha-\beta)}$$

$$= \sqrt{(\alpha,\alpha)-(\alpha,\beta)-(\beta,\alpha)+(\beta,\beta)}$$

$$= \sqrt{(\alpha,\alpha)+(\beta,\beta)} = \sqrt{2}$$

By symmetricity of inner product space.

**44. The rank, index and signature of a cannonical form polynomial $x_1^2-4x_2^2+6x_3^2+2x_1x_2-4x_1x_3+2x_4^2-6x_3x_4$ is**

**विहित( केनोनिकल ) रूप बहुपद $x_1^2-4x_2^2+6x_3^2+2x_1x_2-4x_1x_3+2x_4^2-6x_3x_4$ की कोटि, घातांक एवं चिन्हिका हैं -**

(a) 4, 1, –2 (b) 3, 2, 1
(c) 3, 3, 3 (d) 2, 2, 2

**Ans. (*) :** The given polynomial is:

P(x) = $x_1^2-4x_2^2+6x_3^2+2x_1x_2-4x_1x_3+2x_4^2-6x_3x_4$

The polynomial can be expressed as:

$P(x) = x^TAx$,

Where $x = [x_1, x_2, x_3, x_4]^T$ and A is the symmetric matrix of coefficients. The matrix A is constructed as:

$$A=\begin{bmatrix}1 & 1 & -2 & 0\\ 1 & -4 & 0 & 0\\ -2 & 0 & 6 & -3\\ 0 & 0 & -3 & 2\end{bmatrix}$$

The eigenvalues of A determine the rank, index, and signature.

1. The number of non-zero eigenvalues.
2. The number of negative eigenvalues.
3. The difference between the number of positive and negative eigenvalues.

We calculate the eigenvalues of A to proceed. Let me compute them.

{– 4.21, – 0.34, 1.49, 8.06}

1. The rank is the number of non-zero eigenvalues, are non-zero, so the rank is 4.
2. The index is the number of negative eigenvalues. Here, there are 2 negative eigenvalues (– 4.21, –0.34).
3. The signature is the difference between the number of positive and negative eigenvalues. There are 2 positive eigenvalues (1.49, 8.06) and 2 negative eigenvalues, so the signature is 2 – 2 = 0.

The correct option is 4, 2, 0.

**45. Let the function g(t) > 0 be integrable on [$\alpha$, $\infty$[, and for each fixed x$\in$ I the function h(x,t) is integrable. If $|h(x,t)| \le g(t)$ then the integral $f(x)=\int_a^\infty h(x,t)dt$, is**

**माना कि फलन g(t) > 0 अंतराल [$\alpha$, $\infty$[ पर समाकलनीय है प्रत्येक नियत x$\in$ I के लिए फलन h(x,t) समाकलनीय है। यदि $|h(x,t)| \le g(t)$ हो तो समाकलन $f(x)=\int_a^\infty h(x,t)dt$**

(a) convergent on I / I पर अभिसारी है।
(b) not convergent on I/ I पर अभिसारी नहीं है।
(c) uniformly convergent on I / I पर एकसमान अभिसारी है।
(d) not uniformly convergent on I/ I पर एकसमान अभिसारी नहीं है।

**Ans. (a):** absolutely convergent on I

If $|h(x,t)| \le g(t)$ then

$$\int_a^\infty |h(x,t)|dt \le \int_a^\infty g(t)dt \text{ on I}$$

Now g(t)> 0 is integrable on [a,$\infty$) and for each fixed x $\in$ I the function h(x,t) is integrable then by comparison principle, f(x) = $\int_a^\infty h(x,t)dt$ converges absolutely on I.

**46. If the integral $\int_{-\infty}^{\infty} f(x)dx$ exists as an improper Riemann integral for a non negative function f(x) which increases in (–$\infty$, 0] and decreases in [0, $\infty$), then $\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx$ equals to**

**एक ऋणेतर फलन f(x) जो कि (–$\infty$, 0] में बढ़ता और [0, $\infty$)में घटता हो, के लिए यदि समाकलन $\int_{-\infty}^{\infty} f(x)dx$ का अस्तित्व एक अनंत रीमान समाकलन के जैसा हो, तो $\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx$ के बराबर होगा -**

(a) $\sum_{n=-\infty}^{\infty} f(m^+)+f(m^-)$

(b) $\sum_{n=-\infty}^{\infty} f(m^+)-f(m^-)$

(c) $\sum_{n=-\infty}^{\infty} \frac{f(n^+)+f(n^-)}{2}$

(d) $\sum_{n=-\infty}^{\infty} \frac{f(n^+)-f(n^-)}{2}$

**Ans. (c) :** The given integral involves a non negative function f(x), which satisfies the following property.

1. f(x) increases on (–$\infty$, 0]
2. f(x) decreases on [0, $\infty$)
3. The improper Riemann integral

$$\int_{-\infty}^{\infty} f(x)dx \text{ exists}$$

we are tasked with evaluating

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx$$

the term $\int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx$ corresponding to the Fourier transformation of f(x) denoted by $f(n)$ thus the

$$\sum_{n=-\infty}^{\infty} f(n)$$

where

$$f(n) = \int_{-\infty}^{\infty} f(x).e^{-2\pi inx}dx$$

by applying the Poisson summation

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} f(m)$$

Thus the problem reduces to evaluating the summation

$$\sum_{m=-\infty}^{\infty} f(m)$$

The value of $\sum_{m=-\infty}^{\infty} f(m)$ is determined by the continuity and behavior of f(x) at the integer based on the option provided the correct result is

$$\sum_{n=-\infty}^{\infty} \frac{f(n^+)+f(x^-)}{2}$$

**47. For the function f(x,y) defined as** $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, \text{if}(x,y) \neq 0 \\ 0 \quad, \text{if}(x,y) = 0 \end{cases}$ **at the origin directional derivative**

**फलन f(x,y) जो कि इस प्रकार परिभाषित है -** $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, \text{यदि}(x,y) \neq 0 \\ 0 \quad, \text{यदि}(x,y) = 0 \end{cases}$

**इसके मूल बिन्दु पर दिक् अवकलज का -**

(a) exists and f(x) is continuous. / अस्तित्व है एवं फलन f(x) सतत है।

(b) exists and f(x) is not continuous. / अस्तित्व है एवं फलन f(x) सतत नहीं है।

(c) dose not exists and f(x) is continuous. / अस्तित्व नहीं है एवं फलन f(x) सतत है।

(d) dose not exists and f(x) is not continuous./ अस्तित्व नहीं है एवं फलन f(x) सतत नहीं है।

**Ans. (b) :** exists and f(x) is not continuous.

Let $\vec{u} = (\cos\theta, \sin\theta)$.

Then directional derivative at the origin

$$D_u f(0,0) = \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} \quad (-\infty < t < \infty)$$

$$= \lim_{t\to 0} \frac{t^3 \cos\theta \sin^2\theta}{t^3\cos^2\theta + t^5\sin^4\theta}$$

$$= \lim_{t\to 0} \frac{\cos\theta \sin^2\theta}{\cos^2\theta + t^2\sin^4\theta}$$

$$= \frac{\cos\theta.\sin^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos\theta}$$

Now, if cos θ ≠ 0 then directional derivative at origin exists.

If cosθ = 0 then sinθ ≠ 0 so the $D_u f(0,0) = 0$.

Thus directional derivative at origin exists.

Now, if we put $x = my^2$ and let $y \to 0$, we get

$$\lim_{y\to 0} \frac{my^4}{m^2y^4 + y^4} = \frac{m}{m^2+1}$$

which is different for different value of m.

Hence the $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

Thus, f(x, y) is not continuous at origin.

**48. If T is a continuous linear transformation of Banach space B onto Banach space B' then.**

**यदि T बनाख समष्टि B का बनाख समष्टि B' पर सतत रैखिक रूपांतरण हो तो -**

(a) T is an open mapping./ T एक विवृत प्रतिचित्रण है।

(b) T is a closed mapping. / T एक संवृत प्रतिचित्रण है।

(c) T is open as well as closed mapping. / T विवृत के साथ-साथ संवृत प्रतिचित्रण भी है।

(d) None of these /इनमें से कोई नहीं।

**Ans. (c) :** T is open as well as closed mapping.

Open Mapping Theorem dictates that a linear transformation T of banach space B onto banach space B' must be open.

The graph of a continuous linear transformation T of banach space B onto banach space B', denoted by G(T):

$$G(T) = \{(x,y) | y = T(x)\} \subset B \times B'$$

is closed in B × B'.

**Note :**The official answer key released by the commission says (a).

**49. If A and B are disjoint sets, then**

**यदि A एंव B असंयुक्त समुच्चय है तब -**

(a) $m^*(A\cup B) \leq m^*(A)+m^*(B)$

(b) $m^*(A\cup B) \geq m^*(A)+m^*(B)$

(c) $m^*(A\cup B) = m^*(A)+m^*(B)$

(d) None of these /इनमें से कोई नहीं

**Ans. (c) :** $m^*(A\cup B) = m^*(A)+m^*(B)$

**Definition-** Let m* be an outer measure on a set X. A subset A ⊂ X is Caratheodory measurable with respect to m*, or measurable if

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$$

for every subset E ⊂ X. Thus, a measurable set A splits any set E into disjoint pieces whose outer measures add up to the outer measure of E.

If A is measurable and A ∩ B = ϕ (disjoint), then by taking E = A ∪ B, we see

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

**50. Metric d is known as pseudo metric if -**

**दूरिक d, छदम दूरिक कहलाती है, यदि-**

(a) $d(x, y) = 0 \Rightarrow x = y$

(b) $d(x, y) = 0 \Leftrightarrow x = y$

(c) $x = y \Rightarrow d(x, y) = 0$
(d) none of these / इनमें से कोई नहीं

**Ans. (c) :** $x = y \Rightarrow d(x, y) = 0$
**Difinition-** A pseudometric on a set X is a non-negative real-valued function $d:X\times X\to R$ which satisfies the following conditions
(a) if x = y, then d(x, y) = 0
(b) d(x,y) = d(y,x) (symmetry)
(c) $d(x,y) \le d(x, z) + d(z,y)$ (tringle inequality), where x,y,z are arbitrary elements of X.
In pseudometric it is not required that d(x,y) = 0 implies x = y.

**51. Value of $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{n+n}\right)$ is -**
**$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{n+n}\right)$ का मान होगा -**

(a) $\log\frac{1}{2}$ (b) log 1
(c) log 2 (d) log 4

**Ans. (c) :** log 2
we have $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{n+n}\right)=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$

Now $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right)}\left(\frac{1}{n}\right)=\int_0^1\frac{1}{1+x}dx$

$$= \log(1+x)\Big|_0^1$$
$$= \log 2.$$

**52. Every single valued differentiable function f(z) of complex variable z, in a domain D, satisfy**
**सम्मिश्र चर z का प्रत्येक एकल मान अवकलनीय फलन f(z) ( एक प्रक्षेत्र में ) संतुष्ट करता है-**

(a) Laplace's equation/लाप्लास समीकरण
(b) Legendre's equation /लेगेन्ड्र समीकरण
(c) Laguerr's equation/लेगेर समीकरण
(d) Liouvile's equation/लिओविले समीकरण

**Ans. (a) :** Laplace's equation
Let $f(z) = u(x, y) + iv(x, y);\ \forall z = x + iy \in D \subseteq \mathbf{C}$ be a single valued function of complex variable z. If f(z) is differentiable function then u and v satisfy Cauchy - Riemann equations i.e.

$$u_x = v_y \text{ and } v_x = -u_y$$

where subscripts denote partial derivative.
Now, we have $u_{xx} = v_{yx}$ and $v_{xy} = -u_{yy}$ (partially differentiating)
$\Rightarrow$ Laplace's equation ; $u_{xx} + u_{yy} = 0$ is satisfied.

**53. Zeros of an analytic function f(z) are**
**एक वैश्लेषिक फलन के शून्यक होते है -**

(a) Simple zeros/साधारण शून्यक
(b) Isolated singularities /वियुक्त विचित्रता
(c) non-isolated singularities /अवियुक्त विचित्रता
(d) isolated /वियुक्त

**Ans. (d) :** isolated
**Theorem-** Suppose f is a holomorphic function in a region $\Omega$ that vanishes on a sequence of distinct points with a limit point in $\Omega$. Then f is identically 0.
**Proof-** Suppose that $z_0 \in \Omega$ is a limit point for the sequence $\{w_k\}_{k=1}^{\infty}$ and that $f(w_k) = 0$. First, we show that f is identically zero in a small disc containing $z_0$. Choose a disc D centered at $z_0$ and contained in $\Omega$, and consider the power series expansion of f in that disc

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

If f is not identically zero, there exists a smallest integer m such that $a_m \neq 0$. But then we can write

$$f(z) = a_m(z-z_0)^m.(1+ g(z-z_0))$$

where $g(z-z_0)$ converges to 0 as $z\to z_0$.
Taking $z = w_k \neq z_0$ for a sequence of points converging to $z_0$, we get a contradiction since $a_m(w_k - z_0)^m \neq 0$ and $1+g(w_k - z_0) \neq 0$, but $f(w_k) = 0$.

Let U denote the interior of the set of points where f(z) = 0 Then U is open by definition and non-empty. The set U is also closed since if $z_n \in U$ and $z_n \to z$, then f(z) = 0 by continuity, and f vanishes in a neighborhood of z by the argument above.
Hence $z \in U$. Now if we let V denote the complement of U in $\Omega$, we conclude that U and V are both open , disjoint, and

$$\Omega = U \cup V.$$

since $\Omega$ is connected we conclude that either U or V is empty.
Since $z_0 \in U$, we find that $U = \Omega$ and the proof is complete.
A complex number $z_0$ is a zero for the holomorphic function if $f(z_0) = 0$. In particular, above theorem (analytic continuation) shows that the zeros of a non-trivial holomorphic function are isolated.

**54. A function f(z) has no singularity in the finite part of the plane but has a pole of order m at infinity. then ,**
**किसी समतल के परिमित भाग में एक फलन f(z) की कोई विचित्रता नहीं होती है किन्तु अनन्त पर m कोटि का एक अनन्तक हो तो,**

(a) f(z) is a polynomial of degree m. / f(z), m घात का एक बहुपद है।
(b) f(z) has zero of order m. / f(z), का शून्य m कोटि का है।
(c) f(z) has singularity. / f(z) विचित्रता रखता है ।
(d) None of these/इनमें से कोई नहीं।

**Ans. (a):** f(z) is a polynomial of degree m.
If f is entire then f has a pole of order m at infinity if and only if the function g defined on **C**\{0} by g(z) = $f\left(\frac{1}{z}\right)$ has a pole of order m at 0 i.e. there exists an entire function h such that

$$f\left(\frac{1}{z}\right) = g(z) = \frac{h(z)}{z^m} \text{ and } h(0) \neq 0.$$

Now $|f(z)| = \left|\frac{h\left(\frac{1}{z}\right)}{z^{-m}}\right| \leq M|z|^m$ whenever $|z| \geq 1$

where $M = \max_{|z|\leq 1}|h(z)|$

Now if r > 1 and k > m, then with $\gamma(t) = re^{it}$ we have

$$|f^k(0)| = \left|\frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{w^{k+1}}dw\right|$$

$$= \frac{k!}{2\pi i}\left|\int_0^{2\pi} \frac{f(re^{it})}{(re^{it})^{k+1}} ire^{it}dt\right|$$

$$\leq \frac{k!}{2\pi r^k}\int_0^{2\pi}|f(re^{it})|dt$$

$$\leq \frac{k!}{2\pi r^k} 2\pi \max_{0\leq t\leq 2\pi}|f(re^{it})|$$

$$\leq \frac{k!}{2\pi r^k} 2\pi M r^m$$

$$= \frac{Mk!}{r^{k-m}}$$

Since k > m it follows by letting r → ∞ that $f^k(0) = 0$. Summarizing that if k > m then $f^k(0) = 0$. This implies that the power series representation of f at 0 is in fact a polynomial with degree ≤ m. Now the degree equals m follows from that h(0) ≠ 0.

**55. Value of integral $\int_c \frac{zdz}{(9-z^2)(z+i)}$, where C is the circle $|z| = 2$ (Using Cauchy's integral formula) equals to -**

**समाकल $\int_c \frac{zdz}{(9-z^2)(z+i)}$ का मान, जहाँ कि C वृत्त $|z| = 2$ है, (कॉशी समाकल सूत्र द्वारा) बराबर होगा -**

(a) π/2 (b) π/3
(c) π/5 (d) π/7

**Ans. (c) :** π/5

Given $\int_C \frac{zdz}{(9-z^2)(z+i)}$; C is the circle $|z| = 2$.

Cauchy's integral formula :

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \text{ for any point } a \in D.$$

So, we have

$$f(-i) = \frac{1}{2\pi i}\int_C \frac{zdz}{(9-z^2)(z+i)};\ f(z) = \frac{z}{(9-z^2)}$$

$$\Rightarrow \frac{-2\pi i^2}{(9-(-i)^2)} = \int_C \frac{zdz}{(9-z^2)(z+i)}$$

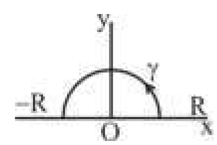
$$\Rightarrow \int_C \frac{zdz}{(9-z^2)(z+i)} = \frac{\pi}{5}$$

**56. Contour integration $\int_0^\infty \frac{\log(1+x^2)}{(1+x^2)}dx$ gives value equals to,**

**परिरेखीय समाकल (कन्टूर समाकल) $\int_0^\infty \frac{\log(1+x^2)}{(1+x^2)}dx$ का मान होता है -**

(a) πlog2 (b) –πlog2
(c) $\frac{\pi}{2}\log 2$ (d) $-\frac{\pi}{2}\log 2$

**Ans. (a) :** πlog2




Let $\int_C \frac{\log(z+i)}{1+z^2}dz = \int_C f(z)dz$

,where C is the contour consisting of a large semi-circle γ of radius R in the upper half of the plane and the part of the real axis from x = –R to x = R.
By residue theorem, we get

$$\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_\gamma f(z)dz = 2\pi i\sum R^+$$

Let, $z = Re^{i\theta}$, we get

$$\left|\int_\gamma f(z)dz\right| \leq \int_0^\pi \left|\frac{\log(Re^{i\theta}+i).Rie^{i\theta}}{1+R^2.e^{2i\theta}}\right|d\theta$$

$$\leq \int_0^\pi \frac{|\log Re^{i\theta}| + \log\left(1+\frac{i}{Re^{i\theta}}\right)}{R^2-1}Rd\theta$$

$$\left[\because |R^2e^{2i\theta}+1| = |R^2e^{2i\theta}-(-1)| \geq |R^2.e^{2i\theta}| - |-1| = R^2-1\right]$$

$$\leq \int_0^\pi \left[\frac{R(\log R+\theta)}{R^2-1} + \frac{R}{R^2-1}\left|\log\left(1+\frac{i}{R}e^{-i\theta}\right)\right|\right]d\theta$$

→ 0 as R → ∞

Now, since $\lim_{R\to\infty}\frac{R\log R}{R^2-1}=\lim_{R\to\infty}\frac{\log R}{R}.\frac{R^2}{R^2-1}=0$

and $\lim_{R\to\infty}\frac{R}{R^2-1}\left|\log(1+\frac{i}{R}e^{-i\theta})\right|=0$

Hence when R→ ∞, we get

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\sum R^+.$$

Now f(z) has a simple pole at z = +i and a logarithmic singularity at z = –i of which z = i lies inside C.

Residue of f(z)(at z = i)

$$=\lim_{z\to i}(z-i)\frac{\log(z+i)}{(z+i)(z-i)}=\frac{\log 2i}{2i}=\frac{1}{2i}\left[\log 2+\frac{i\pi}{2}\right]$$

Therefore $\int_{-\infty}^{\infty}\frac{\log(x+i)}{x^2+1}dx=\pi\left[\log 2+\frac{i\pi}{2}\right]$

Equating real parts, we have

$$\int_{-\infty}^{\infty}\frac{1}{2}\frac{\log(1+x^2)}{1+x^2}dx=\pi\log 2$$

Hence $\int_{0}^{\infty}\frac{\log(1+x^2)}{1+x^2}dx=\pi\log 2$

**57. Value of integration** $\int_0^{2\pi}e^{\cos\theta}.\cos(\sin\theta-n\theta)d\theta=$

**समाकलन का मान -** $\int_0^{2\pi}e^{\cos\theta}.\cos(\sin\theta-n\theta)d\theta=$

(a) $2\pi i/n$ (b) $2\pi/\lfloor n$

(c) $2\pi/n$ (d) $2\pi ni$

**Ans. (b) :** $\frac{2\pi}{n!}$

Consider I = $\int_0^{2\pi}e^{\cos\theta}\cos(\sin\theta-n\theta)d\theta$

= Real part of $\int_0^{2\pi}e^{\cos\theta}.e^{-(n\theta-\sin\theta)i}d\theta$

=Real part of $\int_0^{2\pi}e^{\cos\theta+i\sin\theta}.e^{-n\theta i}d\theta$

=Real part of $\int_0^{2\pi}e^{e^{i\theta}}.e^{-n\theta i}d\theta$ $\left\{\begin{array}{l} z=e^{i\theta}\\ dz=ie^{i\theta}d\theta\\ d\theta=dz/ie^{i\theta}\\ d\theta=dz/iz\\ z^n=e^{ni\theta}\end{array}\right\}$

= Real part of $\frac{1}{i}\int_c\frac{e^z}{z^{n+1}}dz$

= Real part of $\frac{1}{i}\int_c f(z)dz$ ; c = unit circle

Clearly f(z) has a pole of order (n+1) at the origin

The residue of (z) at the origin

$$=\frac{1}{n!}\left[\frac{d^n}{dz^n}.e^z\right]_{z=0}$$

$$=\frac{1}{n!}$$

Hence I = $2\pi i.\frac{1}{i}.\frac{1}{n!}=\frac{2\pi}{n!}$

Equating real and imaginary parts,

$$\int_0^{2\pi}e^{\cos\theta}\cos(\sin\theta-n\theta)d\theta=\frac{2\pi}{n!}$$

and $\int_0^{2\pi}e^{\cos\theta}\sin(n\theta-\sin\theta)d\theta=0$

**58. If an entire function is bounded then it is**
**यदि सर्वत्र वैश्लैषिक फलन परिबद्ध होता है, तब**

(a) $|f(z)|\ge M$

(b) Constant / नियतांक

(c) Analytic/ वैश्लेषिक

(d) None of these/ इनमें से कोई नहीं

**Ans. (b) :** Constant.

**Liouville's Theorem**- Let f be an entire function that is bounded then f is a constant function.

**Proof :**f is bounded i.e. there is a real number M ≥ 0 such that $|f(z)|\le M$ for all z belonging to domain of f.

Let a, b∈**C**. Consider γ(0,R) for a large R. By Cauchy Integral formula, we have

$$|f(a)-f(b)|=\left|\frac{1}{2\pi i}\int_{\gamma(0,R)}\frac{f(z)}{z-a}dz-\frac{1}{2\pi i}\int_{\gamma(0,R)}\frac{f(z)}{z-b}dz\right|$$

assuming a, b is in interior of γ(0,R). We have

$$|z-a|\ge R-|a|;\forall z\in\gamma(0,R)^*$$

$$|z-b|\ge R-|b|;\forall z\in\gamma(0,R)^*$$

Now, $|f(a)-f(b)|=\left|\frac{1}{2\pi i}\int_{\gamma(0,R)}\frac{f(z)(a-b)}{(z-a)(z-b)}dz\right|$

$$\le\frac{1}{2\pi}\int_{\gamma(0,R)}\frac{|f(z)||a-b|}{|z-a||z-b|}|dz|$$

$$\le\frac{1}{2\pi}M|a-b|\int_{\gamma(0,R)}\frac{1}{|z-a||z-b|}|dz|$$

$$\le\frac{1}{2\pi}M|a-b|\frac{1}{(R-|a|)(R-|b|)}\int_{\gamma(0,R)}|dz|$$

$$=\frac{1}{2\pi}M|a-b|\frac{1}{(R-|a|)(R-|b|)}2\pi R$$

$$=M|a-b|\frac{1}{\left(1-\frac{|a|}{R}\right)(R-|b|)}$$

$$\therefore |f(a)-f(b)|\le M|a-b|\frac{1}{\left(1-\frac{|a|}{R}\right)(R-|b|)}$$

As $R \to \infty \left(1-\frac{|a|}{R}\right) \to 1 \,\&\, \left(R-|b|\right) \to \infty$

So $|f(a)-f(b)|$ becomes arbitrarily small & hence

f(a) = f(b) i.e. f is a constant function.

**59. If (X, τ) is a topological space, then which one is not true?**
**यदि (X, τ) एक सांस्थितिक समष्टि है, तब निम्नलिखित में से कौन सा सही नहीं है ?**

(a) A complete regular space is regular./एक पूर्ण सम समष्टिसम होती है ।
(b) Every subspace of a $T_2$– space is a $T_1$– space./ $T_2$– समष्टि की प्रत्येक उपसमष्टि एक $T_1$– समष्टि है ।
(c) Every metric space is a $T_1$– space./प्रत्येक दूरिक समष्टि एक $T_1$– समष्टि है।
(d) Every metric space is a Housdorff space./प्रत्येक दूरिक समष्टि एक हाउसडॉर्फ समष्टि है।

**Ans. (a) :**
Let X be a topological space. Suppose that one-point sets are closed in X. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from {x}, there exist disjoint open sets containing x and B respectively.
A space X is **completely regular** if one-point sets are closed in X and if for each point $x_0$ and each closed set A not containing $x_0$, there is a continuous function
$f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.
**A completely regular space is regular**, since for given f, the sets

$f^{-1}\left(\left[0,\frac{1}{2}\right]\right)$ and $f^{-1}\left(\left[\frac{1}{2},1\right]\right)$ are disjoint open sets about A and $x_0$.

**Every $T_2$- space (Hausdorff space) is $T_1$- space**
From the definition of $T_2$ (Hausdorff) space-
$\forall x, y \in X : x \neq y : \exists A, B \in \tau : x \in A, y \in B : A \cap B = \phi$
so $\exists A \in \tau : x \in A, y \notin A$ and $\exists B \in \tau : y \in B, x \notin B.$
which is precisely the characterization of $T_1$-space.
Because every subspace of a Hausdorff space is Hausdorff, we have every subspace of Hausdorff space is $T_1$ - space.
**Every metric space is a Hausdorff space** because If x and y are distinct points of the metric space (X,d), we let $\varepsilon = \frac{1}{2}d(x,y)$; then the triangle inequality implies that $B_d(x,\varepsilon)$ and $B_d(y,\varepsilon)$ are disjoint and hence every metric space is $T_1$-space.
Not every regular space is completely regular.
A topological space is regular if every closed subset and a point outside of that subset have non-overlapping open neighborhoods.
A completely regular space is a type of topological space.

**60. Any infinite subset A of a discrete topological space X is,/वियुक्त सांस्थितिक समष्टि X का कोई भी अनन्त उप समुच्चय A है।**

(a) not compact /असंहत
(b) compact /संहत
(c) compact and connected /संहत एवं संबद्ध
(d) connected only / केवल संबद्ध

**Ans. (a) :** not compact.
Take the cover U = {{x}: x ∈ X}
This is clearly an open cover of X because the topology is discrete and all subsets are open and every p∈ X is {p}∈ μ.
A finite subcover is a finite subset of U that together cover X too but if F is a finite subset of U then it consists of finitely many singletons $\{x_1\},\{x_2\},...\{x_N\}$ for some finite N. But X has infinitely many points so there are infinitely many $x \notin \{x_1, x_2 ...... x_N\}$, say p. Then p is in none of the sets of F,and so this finite subset of U is not a cover of X.

**61. The radius of curvature for the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$ is**
**वक्र $x = 3t, y = 3t^2, z = 2t^3$ के लिए वक्रता त्रिज्या है:**

(a) $\frac{3}{2}(1+2t^2)^2$ (b) $\frac{1}{2}(1+2t^2)$
(c) $\frac{1}{2}(1+3t^2)$ (d) $\frac{3}{2}(1+t^2)^2$

**Ans. (a) :** $\frac{3}{2}(1+2t^2)^2$
Given curve
$r = (3t, 3t^2, 2t^3)$
so, $\dot{r} = (3, 6t, 6t^2)$
& $\ddot{r} = (0, 6, 12t)$
Now $\dot{r} \times \ddot{r} = 18\left(2t^2\hat{i} - 2t\hat{j} + 1\hat{k}\right)$

Now; curvature $k = \frac{\|\dot{r}\times\ddot{r}\|}{\|\dot{r}\|^3}$

$$= \frac{18(1+2t^2)}{27(1+2t^2)^3} = \frac{2}{3(1+2t^2)^2}$$

∴ Radius of curvature $\rho = \frac{1}{k} = \frac{3}{2}(1+2t^2)^2$.

**62. The principal radii at the origin of the surface $2z = 5x^2 + 4xy + 2y^2$ is**
**सतह $2z = 5x^2 + 4xy + 2y^2$ की मूल बिन्दु पर प्रमुख त्रिज्याएं हैं –**

(a) $1, \frac{1}{6}$ (b) $3, \frac{1}{2}$
(c) $2, \frac{1}{6}$ (d) $1, \frac{1}{2}$

**Ans. (a) :** $1, \frac{1}{6}$

Given $2z = 5x^2 + 4xy + 2y^2$

Now $\frac{\partial z}{\partial x} = z_1 = 5x + 2y$

$\frac{\partial z}{\partial y} = z_2 = 2x + 2y$

$\frac{\partial^2 z}{\partial x^2} = 5, \& \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 2$

At the origin

$$\frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial y} = 0, \frac{\partial^2 z}{\partial x^2} = 5, \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y^2} = 2$$

Now, $E = 1 + \left(\frac{\partial z}{\partial x}\right)^2 = 1, F = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 0$

$G = 1 + \left(\frac{\partial z}{\partial y}\right)^2 = 1, \ H = 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$

So $L = \frac{\frac{\partial^2 z}{\partial x^2}}{H} = 5, M = \frac{\frac{\partial^2 z}{\partial y^2}}{H} = 2 \ N = \frac{\frac{\partial^2 z}{\partial y \partial x}}{H} = 2$

and equation of principal curvature

$$H^2 k_n^2 - K_n (EN - 2FM + LG) + LN - M^2 = 0$$

gives $k_n^2 - 7k_n + 6 = 0$

$\Rightarrow k_n = 1, 6$

$\Rightarrow$ Principal radii $= 1, \frac{1}{6}$

**63. The geodesic on a right circular cylinder is :**
**लंब वृत्तीय बेलन पर अल्पांतरी है-**

(a) Circle /वृत्त
(b) Helix/कुंडलिनी (हेलिक्स)
(c) Line/रेखा
(d) Ellipse/दीर्घवृत्त

**Ans. (b) :** Helix

**Proposition-** The geodesics on a right circular cylinder are helices.

**Proof-** If the surface of a revolution is a right circular cylinder, the meridian are generators and the distance between the meridian and the axis of the cylinder is a constant a (say).

Hence if $\psi$ is the angle between the geodesic and the generator, then by Clairaut's theorem a $\sin\psi = h$ where h is a constant.

Thus $\sin\psi = \frac{h}{a}$ so that $\psi$ is a constant. Hence a geodesic on a right circular cylinder cuts the generators at a constant angle and therefore it is a helix.

**64. The length of the curve given as the intersection of the surfaces $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x = a \cosh\left(\frac{z}{a}\right)$ from the point (a,0,0) to the point (x,y,z).**

**दिए गए सतहों $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x = a \cosh\left(\frac{z}{a}\right)$ के प्रतिच्छेदन पर, बिन्दु (a,0,0)से बिन्दु (x,y,z) तक वक्र की लंम्बाई है -**

(a) $\frac{y\sqrt{a^2+b^2}}{b}$
(b) $\frac{x\sqrt{a^2+b^2}}{b}$
(c) $\frac{y\sqrt{a^2+b^2}}{a}$
(d) $\frac{x\sqrt{a^2+b^2}}{a}$

**Ans. (a) :** $\frac{y\sqrt{a^2+b^2}}{b}$

We have $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and $x = a\cosh\frac{z}{a}$ $\left(t = \frac{z}{a}\right)$

Let $x = a\cosh t, \ y = b\sinh t, \ z = at$

Clearly both the curves are satisfied.

Let r be the position vector of the point (x, y, z) on the given curve

$\therefore \quad r = (a\cosh t)\hat{i} + (b\sinh t)\hat{j} + at\hat{k}$

At the point (a, 0, 0) we have

$a\cosh t = a, \ b\sinh t = 0, \ at = 0 \Rightarrow t = 0$

$\dot{r} = (a\sinh t)\hat{i} + (b\cosh t)\hat{j} + a\hat{k}$

$|\dot{r}| = \sqrt{a^2\sinh^2 t + b^2\cosh^2 t + a^2}$

$= \sqrt{a^2(\sinh^2 t + 1) + b^2\cosh^2 t}$

$= \sqrt{a^2\cosh^2 t + b^2\cosh^2 t} = \sqrt{(a^2+b^2)}\cosh t$

$\therefore$ Length of the given curve

$= \int_0^t |\dot{r}| dt = \int_0^t \sqrt{a^2+b^2}\cosh t \, dt$

$= \sqrt{a^2+b^2}\sinh t\Big|_0^t = \sqrt{a^2+b^2}\sinh t$

$= \frac{\sqrt{a^2+b^2}}{b} y$

**65. A subset Y of a topological space (X, $\tau$) is no where dense if -**
**किसी सांस्थितिक समष्टि (X, $\tau$) का कोई उपसमुच्चय Y कहीं पर भी सघन नहीं होता है , यदि-**

(a) Y has empty interior/ Y का रिक्त अभ्यंतर हो
(b) $\bar{Y}$ has empty interior/ $\bar{Y}$ का रिक्त अभ्यंतर हो

(c) Closure of interior Y is non empty /अभ्यंतर Y का संवरक अरिक्त हो
(d) None of these /इनमें से कोई नहीं

**Ans. (b) :** $\bar{Y}$ has empty interior.
Let Y be a subset of a topological space (X, τ) then $\bar{Y}$ is the closure of Y.
**Definition**- A subset Y of a topological space (X, τ) is nowhere-dense in X if $\bar{Y}$ has empty interior.

**66. How many linearly independent solution may exist, of a linear homogeneous equation of order n ?**
**n कोटि के रैखिक समघात समीकरण के कितने रैखिकतः स्वतंत्र हल अस्तित्व में हो सकतें हैं ?**
(a) More than n/ n से अधिक
(b) n +1 only / n +1 केवल
(c) n – 1 only / n – 1केवल
(d) less than or equals to n/से कम अथवा बराबर

**Ans. (d) :** Less than or equal to n.
Linear equation of order n
$\left[D^n + b_1(x)D_{n-1} + ...... + b_{n-1}(x)D + b_n(x)\right]y = R(x)$;
$\frac{d}{dx} \equiv D$.
The corresponding homogeneous equation is
$$\left[D^n + b_1(x)D^{n-1} + .... + b_{n-1}(x)D + b_n(x)\right]y = 0$$
There may exist at most n linearly independent solution of above homogeneous equation.

**67. Singular solution of the differential equation $(8p^3 - 27)x = 12p^2y$ is**
**अवकलन समीकरण $(8p^3 - 27)x = 12p^2y$ का विचित्र हल है-**
(a) $4y^3 + x^3 = 0$
(b) $4y^3 + 27x^3 = 0$
(c) $4y^3 - 27x^3 = 0$
(d) $4y^3 - x^3 = 0$

**Ans. (b) :** $4y^3 + 27x^3 = 0$
The given equation $(8p^3-27)x = 12p^2y$ ; $p = \frac{dy}{dx}$
$$y = \frac{2}{3}px - \frac{9}{4}\left(\frac{x}{p^2}\right)$$
On differentiating; we get
$$p = \frac{2}{3}\left(p + x\frac{dp}{dx}\right) - \frac{9}{4}\left(\frac{1}{p^2} - \frac{2x}{p^3}\frac{dp}{dx}\right)$$
or $\frac{1}{3}p + \frac{9}{4p^2} - x\frac{dp}{dx}\left(\frac{2}{3} + \frac{9}{2p^3}\frac{dp}{dx}\right) = 0$
or $\frac{1}{3}p\left(1 + \frac{27}{4p^3}\right) - \frac{2x}{3}\frac{dp}{dx}\left(1 + \frac{27}{4p^3}\right) = 0$
or $\frac{1}{3}\left(1 + \frac{27}{4p^3}\right)\left(p - 2x\frac{dp}{dx}\right) = 0$

Omitting the first factor which does not involve $\frac{dp}{dx}$, we get $p - 2x\frac{dp}{dx} = 0$ or $\frac{2}{p}dp = \frac{1}{x}dx$.
Integrating, we get
$$2\log p = \log x + \log\left(\frac{9}{4c}\right)$$
$$\text{or } p^2 = x\left(\frac{9}{4c}\right)$$
$$\text{or } p = \pm\frac{3}{2}\left(\frac{x}{c}\right)^{\frac{1}{2}}$$
Now putting this value of p in the equation required general solution is
$$y = \pm\left(\frac{x^{\frac{3}{2}}}{c^{\frac{1}{2}}}\right) - c$$
or $c^{\frac{1}{2}}(y + c) = \pm x^{\frac{3}{2}}$
or $c(y + c)^2 = x^3$
Differentiating w.r.t. c ;
$(y + c)^2 + 2c(y + c) = 0$
or $(y + c)(y + 3c) = 0$
$\Rightarrow y + c = 0$
or $y + 3c = 0$
$\Rightarrow c = -y$ or $c = -\frac{y}{3}$
When $c = -y$ we have $x^3 = 0$ or $x = 0$
when $c = \frac{-y}{3}$ we have $4y^3 + 27x^3 = 0$
These are the required singular solutions.

**68. The general solution of the Partial Differential Equation $(y + zx)p - (x + yz)q = x^2 - y^2$ is**
**अंशिक अवकल समीकरण $(y + zx)p - (x + yz)q = x^2 - y^2$ का व्यापक हल है**
(a) $\phi(x^2 + y^2 + z^2, xy + z) = 0$
(b) $\phi(x^2 + y^2 - z^2, xy + z) = 0$
(c) $\phi(x^2 + y^2 + z^2, xyz) = 0$
(d) $\phi(xy + yz + zx, xyz) = 0$

**Ans. (b) :** $\phi(x^2 + y^2 - z^2, xy + z) = 0$
Given $(y + zx)p - (x + yz)q = x^2 - y^2$
Here the Lagrange's auxiliary equations are
$$\frac{dx}{y + zx} = \frac{dy}{-(x + yz)} = \frac{dz}{x^2 - y^2}$$

Choosing x,y, –z as multipliers, we get

$$\frac{xdx + ydy - zdz}{x(y+zx) - y(x+yz) - z(x^2 - y^2)} = \frac{xdx + ydy - zdz}{0}$$

$\therefore$ xdx + ydy + (–zdz) = 0

$\Rightarrow$ 2xdx + 2ydy – 2zdz = 0

Integrating, we get

$\Rightarrow$ $x^2 + y^2 - z^2 = c_1$, $c_1$- arbitrary constant.

Now choosing y, x, 1 as multipliers, we get

$$\frac{ydx + xdy + dz}{y(y+zx) - x(x+yz) + x^2 - y^2} = \frac{ydx + xdy + dz}{0}$$

$\therefore$ ydx + xdy + dz = 0

or d(xy) + dz = 0

Integrating, we get

$xy + z = c_2$; $c_2$ - arbitrary constant

$\therefore$ The required solution is

$\phi(x^2 + y^2 - z^2, xy + z) = 0$, $\phi$ being an arbitrary function.

**69. The complete integral of $q = (z + px)^2$ is**
**$q = (z + px)^2$ का पूर्ण समाकल है -**

(a) $xy = 2\sqrt{a}\sqrt{x} + ax + b$

(b) $xy = 2\sqrt{a}\sqrt{x} + ay + b$

(c) $yz = 2\sqrt{a}\sqrt{y} + ay + b$

(d) $xz = 2\sqrt{a}\sqrt{x} + ay + b$

**Ans. (d) :** $xz = 2\sqrt{a}\sqrt{x} + ay + b$

Given $q = (z + px)^2 \Rightarrow f(x,y,z,p,q) = (z + px)^2 - q = 0$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = -\frac{dy}{f_q}$$

or
$$\frac{dp}{2p(z+px) + 2p(z+px)} = \frac{dq}{2q(z+px)}$$
$$\frac{dz}{-2px(z+px) + q} = \frac{dx}{-2x(z+px)} = \frac{dy}{-1}$$

Taking the second and fourth fractions $\frac{1}{q}dq = -\frac{1}{x}dx$

Integrating, logq = loga –log x $\Rightarrow q = \frac{a}{x}$

substituting this value of q in equation ; we have

$(z + px)^2 = \frac{a}{x}$

or $px = \frac{\sqrt{a}}{\sqrt{x}} - z$

or $p = \frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}$

$\therefore$ $dz = pdx + qdy = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right)dx + \frac{a}{x}dy$

or $xdz = \sqrt{a}x^{-1/2}dx - zdx + ady$

or $xdz + zdx = \sqrt{a}^{-1/2}dx + ady$

or $d(xz) = 2\sqrt{a}\sqrt{x} + ay + b$; a and b being arbitrary constants is the required solution.

**70. The singular solution of the differential equation $p^3 - 4xyp + 8y^2 = 0$**
**अवकल समीकरण $p^3 - 4xyp + 8y^2 = 0$ का विचित्र हल है -**

(a) $4y^3 = 27x$ (b) $27y^3 = 4x$

(c) $27y = 4x^3$ (d) $4y = 27x^3$

**Ans. (c) :** $27y = 4x^3$

Given $p^3 - 4xyp + 8y^2 = 0$

Solving for x, we get $x = \frac{p^3 + 8y^2}{4yp} = \frac{p^2}{4y} + \frac{2y}{p}$

Differentiating w.r.t. y we get

$$\frac{1}{p} = \frac{p}{2y}\frac{dp}{dy} - \frac{p^2}{4y^2} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

or $\frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = \frac{p^2}{4y^2} - \frac{1}{p}$

i.e. $\frac{dp}{dy} = \frac{p}{2y}$

in which variables are separable. The solution is

$p^2 = cy$

Eliminating p between this and the original differential equation we get

$$c^{\frac{3}{2}}y^{\frac{3}{2}} - 4c^{\frac{1}{2}}xy^{\frac{3}{2}} + 8y^2 = 0$$

or $\frac{1}{2}c^{\frac{1}{2}}\left(\frac{1}{4}c - x\right) = -y^{\frac{1}{2}}$

Hence, $y = c(x-c)^2$ ; c – arbitrary constant

Now differentiating w.r.t c we have

$(c - x)^2 + 2c(c - x) = 0$

$\Rightarrow$ $(c - x)(3c - x) = 0$

$\Rightarrow$ $c = x$ or $c = \frac{x}{3}$

Now, when c = x we have

y = 0

and when $c = \frac{x}{3}$ we have

$$y = \frac{x}{3}\left(x - \frac{x}{3}\right)^2$$

or $y = \frac{x}{3}\left(\frac{2x}{3}\right)^2$

or $27y = 4x^3$

# Uttar Pradesh Higher Education Service Commission
# Assistant Professor 2014
## Solved Paper [Exam Date : 07.12.2014]

**1. The principal value of $\log\left(-1+i\sqrt{3}\right)$ is:**

**$\log\left(-1+i\sqrt{3}\right)$ का मूल मान है:**

(a) $\log 2 - i\dfrac{\pi}{6}$ (b) $\log 2 - i\dfrac{2\pi}{6}$

(c) $\log 2 + i\dfrac{2\pi}{3}$ (d) $\log 2 + i\dfrac{\pi}{6}$

**Ans. (c) :** Principal value of log(–1+ i $\sqrt{3}$ ) is given by

$\log(-1+i\sqrt{3}) = \log\left|-1+i\sqrt{3}\right| + i\arg(-1+i\sqrt{3})$

$=\log 2 + i\dfrac{2\pi}{3}$

**2. Let ν be the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Then $\int_\nu xdz$ =:**

**माना कि ν एक दीर्घवृत्त $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ है, तो $\int_\nu xdz$ =:**

(a) iπab (b) πab

(c) ab (d) 0

**Ans. (a) :** Given ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is represented by

z (θ) = a cosθ + i b sinθ, 0 ≤ θ ≤ 2π

Hence $\dfrac{dz}{d\theta} = -a\sin\theta + ib\cos\theta$

So, the given integral becomes

$$I=\int_0^{2\pi} \mathrm{Re}(z(\theta))(-a\sin\theta+ib\cos\theta)d\theta$$

$$I=\int_0^{2\pi} a\cos\theta(-a\sin\theta+ib\cos\theta)d\theta$$

$$\therefore\ I=-a^2\int_0^{2\pi}\cos\theta\sin\theta d\theta + i\int_0^{2\pi} ab\cos^2\theta d\theta$$

$$=-a^2\int_0^0 tdt + \frac{iab}{2}\int_0^{2\pi}(\cos 2\theta+1)d\theta \quad \begin{Bmatrix}\sin\theta=t\\ \cos\theta d\theta = dt\end{Bmatrix}$$

(Substituting sinθ = t in first integral.)

$= 0 + i\pi ab$

$= i\pi ab.$

**3. "An entire function which is bounded on C has to be constant", what is this well known result called?**

**''एक सम्पूर्ण फलन जो कि C पर परिबद्ध है, अचर होगा'', इस सुप्रसिद्ध परिणाम को क्या कहते हैं?**

(a) Cauchy's Therorem/कोशी प्रमेय

(b) Liouville Theorem/लियोविल्ले प्रमेय

(c) Morera's Theorem/मोररा प्रमेय

(d) Taylor's Theorem/टेलर प्रमेय

**Ans. (b) :** Liouville's Theorem:- Let f be an entire function that is bounded on C. Then f is a constant function.

**4. What is the radius of convergence of the power series $\sum_{n=1}^{\infty}(3+4i)^n z^n$?**

**घातीय श्रेणी $\sum_{n=1}^{\infty}(3+4i)^n z^n$ की अभिसारिता की त्रिज्या क्या हैं?**

(a) 5 (b) 0

(c) 1/5 (d) 1/4

**Ans. (c) :** Here $a_n = (3+4i)^n$

Radius of convergence

$$R=\frac{1}{\lim\limits_{n\to\infty}\left|\dfrac{a_n+1}{a_n}\right|}$$

$$=\frac{1}{\lim\limits_{n\to\infty}|(3+4i)|}$$

= 1/5

**5. Which of the following statements is not true? For $z\in C$, if $f(z)=\dfrac{e^z}{e^z-1}$,**

**निम्नलिखित में से कौन सा कथन सत्य नहीं है? $z\in C$ के लिए, यदि $f(z)=\dfrac{e^z}{e^z-1}$, तो:**

(a) f is entirely analytic./ f सर्वत्र वैश्लेषिक है।

(b) The only singularities of f are poles./ f की एकमात्र विचित्रताएँ अनन्तक हैं।

(c) f has infinitely many poles on the imaginary axis./अधिकल्पित अक्ष पर f के अनंततः बहुत अनन्तक है।

(d) each pole of f is simple./ f का हर एक अनन्तक एकघात है।

**Ans. (a):** Here $f(z)=\dfrac{e^z}{e^z-1}$; $z\in C$ to be analytic at z= $2n\pi i$; n=0, ±1,±2.... because denominator $e^z-1$ vanishes only where $e^z=1=e^{i2n\pi i}$ which gives $z=2n\pi i; n=0,\pm1,\pm2,....$

Hence, f is not an entire function, only singularities of f are poles, each pole is simple and f has infinitely many poles on the imaginary axis.

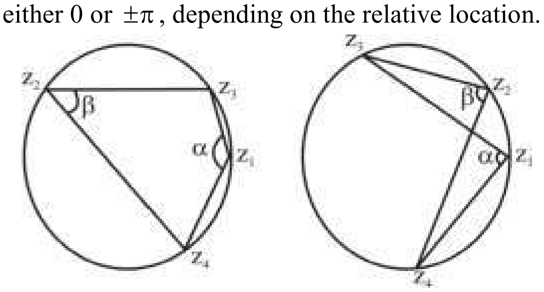
**6. When is the cross-ratio of four points in C real?**
**C में चार बिन्दुओं का तिर्यक अनुपात, वास्तविक कब होगा?**

(a) two of them lie on a circle/उनमें से दो वृत्त पर हों
(b) three of them lie on a circle/उनमें से तीन वृत्त पर हो
(c) all the four lie on a circle/सभी चारों वृत्त पर हों
(d) one of them lies on a circle/उनमें से एक वृत्त पर हो

**Ans. (c) :** The cross ratio $(z_1,z_2,z_3,z_4)$ is real if and only if the four points lie on a circle or on a straight line for we obtain

$$\arg(z_1,z_2,z_3,z_4)=\arg\frac{z_1-z_3}{z_1-z_4}-\arg\frac{z_2-z_3}{z_2-z_4}$$

and if the points lie on a circle this difference of angles in either 0 or $\pm\pi$, depending on the relative location.




**7. Where do all the zeros of $z^4+6z+3=0$ lie?**
**$z^4+6z+3=0$ के सारे शून्य कहाँ होंगे?**

(a) $|z|<2$ (b) $|z|<1$
(c) $1<|z|<2$ (d) $|z|>2$

**Ans. (c) :** Let find the range of |z| for the roots of $z^4+6z+3=0$ mathematically.
Let |z| = r. For a root z, we have:
$|z^4|=|6z+3|\le|6z|+|3|=6r+3.$
thus:
$r^4\le 6r+3$
Define:
$f(r)=r^4-6r-3$
we solve f(r) = 0 for r > 0
Behavior of f(r):
for r = 0, f(0) = – 3 < 0
As $r\to\infty, f(r)\to\infty$.
critical points: Differentiate f(r)
$f'(r)=4r^3-6$
Solve f '(r) = 0
$r^3=\dfrac{3}{2}$
For r > 0, f(r) changes sign near
$r\approx1.2$ and $r\approx2$
Numerical or graphical analysis of
f(r) = 0 confirms:
1 < r < 2
The roots satisfy 1 < |z| < 2

**8. Which of the following statements is not true?**
**निम्नलिखित में से कौन सा कथन सत्य नहीं है?**

(a) Zeros of an analytic functions are isolated./विश्लेषिक फलन के शून्यक आइसोलेटेड होते हैं।
(b) If an analytic function vanishes on a set with a limit point. then it is identically zero./यदि विश्लेषिक फलन ऐसे समुच्चय पर, जिसका एक सीमा बिन्दु हो, शून्य हो वह फलन स्वयं ही शून्य होता है।
(c) Poles are isolated/अनन्तक आइसोलेटेड हैं।
(d) A non-zero analytic function can have zeros of infinite order. /शून्येतर विश्लेषक फलन के अनंत कोटि के शून्यक हो सकते हैं।

**Ans. (d) :** If f(z) is analytic in a region $\Omega$ containing a , then

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+...+\frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1}+\frac{f_n(a)}{n!}(z-a)^n$$

where $f_n(z)$ is analytic in $\Omega$
If f(a) and all derivative $f^{(n)}(a)$ vanish then

$$f(z)=f_n(z)(z-a)^n$$

for any n. Draw a circle C of radius R about a so that C and its inside are contained in Ω. The absolute value $|f(z)|$ has a maximum M on C; we find

$$|f_n(z)|\le\frac{M}{R^{n-1}(R-|z-a|)}$$

for $|z-a|<R.$ Thus we have

$$|f(z)|\le\left(\frac{|z-a|}{R}\right)^n\frac{MR}{R-|z-a|}$$

But $\left(\dfrac{|z-a|}{R}\right)^n\to0$ for $n\to\infty$, since $|z-a|<R.$ Here f (z)=0 inside of C.

We show now that f(z) is identically zero in all of Ω. Let $E_1$ be the set on which f(z) and all derivatives vanish and $E_2$ the set on which the function or one of the derivatives is different from zero. $E_1$ is open by the above reasoning and $E_2$ is open because the function and all derivatives are continuous. Therefore either $E_1$ or $E_2$ must be empty. If $E_2$ is empty, the function is identically zero. If $E_1$ is empty f(z) can never vanish together with all its derivatives.

Assume that f(z) is not identically zero. Then, If f(a) = 0 ,there exists a first $f^h(a)$ which is different from zero, then a is a zero of order h and hence there are no zeros of infinite order.
In this context, an analytic function has the same local behavior as a polynomial, and so we can write

$$f(z) = (z-a)^h f_h(z)$$

where $f_h(z)$ is analytic and $f_h(a) \neq 0$.
Since, $f_h(z)$ is continuous, $f_h(z) \neq 0$ in a neighborhood of a and z = a is the only zero of f(z) in the neighborhood so, the zeros of an analytic function which does not vanish identically are isolated.
If $\lim_{x\to a} f(z) = \infty$, the point a is said to be a pole of f(z), and we get f(a)= ∞. More precisely, to every $a \in \Omega$ there shall exist a neighborhood $|z-a| < \delta$, contained in $\Omega$, such that either f(z) is analytic in the whole neighborhood, or else f(z) is analytic for $0 < |z-a| < \delta$, and the isolated singularity is a pole.
Only zero function can have zeros of infinite order.

**9. Two norms $\|.\|_1$ and $\|.\|_2$ on the same vector space X are equivalent if and only if there are positive real numbers $\alpha$ and $\beta$ such that for $x \in X$:**
**किसी सदिश आकाश X पर दो प्रमाप (norms) $\|.\|_1$ and $\|.\|_2$ समतुल्य होंगे यदि और केवल यदि दो धनात्मक वास्तविक संख्याएँ $\alpha$ और $\beta$ इस प्रकार हैं कि $x \in X$: के लिए:**

(a) $\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_1$ (b) $\alpha\|x\|_2 \le \|x\|_2 \le \beta\|x\|_1$
(c) $\alpha\|x\|_1 \le \|x\|_2 \le \beta\|x\|_2$ (d) $\alpha\|x\|_2 \le \|x\|_1 \le \beta\|x\|_1$

**Ans. (a) :** Two norms defined on the same vector space are said to be equivalent if the topologies induced by these two norms coincide.
The topologies induced by the two norms $\|.\|_{(1)}$ and $\|.\|_{(2)}$ on a vector space X coincide if, and only if, the identity mapping

$$I : \{X, \|.\|_{(1)}\} \to \{X, \|.\|_{(2)}\}$$

is an isomorphism i.e. there exist two constants $K_1 > 0$ and $K_2 > 0$ such that

$$\|x\|_{(2)} \le K_2\|x\|_{(1)} \text{ and } \|x\|_{(1)} \le K_1\|x\|_{(2)}$$

for all $x \in X$. On setting $\alpha = K_1^{-1}$ and $\beta = K_2$ we get that two norms are equivalent if, and only if, there exist two constants $\alpha > 0$ and $\beta > 0$ such that, for all $x \in X$,

$$\alpha\|x\|_{(1)} \le \|x\|_{(2)} \le \beta\|x\|_{(1)}.$$

**10. Which of the following is true?**
**निम्नलिखित में से कौन-सा कथन सत्य हैं?**

(a) An inner product space is a Banach space /एक आन्तर गुणन आकाश बनाक आकाश होता हैं।
(b) An inner product space is a Hilbert space /एक आनतर गुणन आकाश हिल्बर्ट आकाश होता है।
(c) A Hilbert space is a complete inner product space/एक हिल्बर्ट आकाश एक पूर्ण आन्तर गुणन आकाश हैं।
(d) Every Banach space is a Hilbert space /प्रत्येक बनाक आकाश एक हिल्बर्ट आकाश हैं।

**Ans. (c) :** An inner product space is a normed vector space.
**Definition-** A Hilbert space is a Banach space whose norm comes from an inner product.

**11. Which of the following statemnets is true?**
**निम्नलिखित में से कौन-सा कथन सत्य हैं?**

(a) $l_2$ space is not a topological space. /$l_2$आकाश एक संस्थितीय (topological) आकाश नहीं है
(b) $l_p$, $p \neq 2$ is a Hilbert space./$l_p$ $p \neq 2$ एक हिल्बर्ट आकाश है।
(c) $l_p$, $p \neq 2$ is not a Banach space./$l_p$, $p \neq 2$ बनाक आकाश नहीं है।
(d) $l_2$ space is Hilbert space./ $l_2$आकाश एक हिल्बर्ट आकाश हैं।

**Ans. (d) :** Consider sets of real (or complex) sequences

$$x=(x_1, x_2, x_3, ..., x_i, .....)$$

Let $1 \le p < \infty$. we define the space

$$l_p = \left\{ x \left| \sum_{i=1}^{\infty} |x_i|^p < \infty \right. \right\}$$

Then $l_p$ is a Banach space.
Now, consider the space $l_2$ for x and y $\in l_2$, define

$$(x,y) = \sum_{i=1}^{\infty} x_i y_i$$

Where $x = (x_i)$ and $y = (y_i)$ are real sequences. Again, if the base field is **C**, then we define

$$(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

This makes $l_2$ into a Hilbert space.

**12. The norm of an identity operator is:**
**अभिन्न (identity) संकारक का प्रमाप हैं:**

(a) 1 (b) 0
(c) 10 (d) 2

**Ans. (a) :** The norm of the identity operator, $I_x : X \to X$ is 1 i.e. $\|I_x\| = 1$.

**13. Let X and Y be normed spaces over the same field F and T:X→Y is a bounded linear operator. Let N(T) denote the null space Then which of the following statements is correct?**
**मान लिया कि X और Y एक ही क्षेत्र F पर प्रमापीय (normed) आकाश हैं तथा T:X→Y एक बंधित रेखीय संकारक है। मान लिया कि N(T) शून्य आकाश को निरूपित करता है। तो निम्नलिखित में से कौन सा कथन सत्य है?**

(a) $N(T) = \{\phi\}$
(b) $N(T)$ is an open set/ $N(T)$ एक खुला समुच्चय है
(c) $N(T) = X$
(d) $N(T)$ is closed set/ $N(T)$ एक बन्द समुच्चय है

**Ans. (d) :** The null space N(T) is closed.
Proof- Let $x \in \overline{N(T)} \subseteq X$ and a sequence $x_n \in N(T)$ with $x_n \to x$ as $n \to \infty$.
We have $0 = T(x_n)$. Taking $n \to \infty$ and using $\|T(x_n) - T(x)\| \le \|T\|\|x_n - x\|$, we get $0 = T(x)$ and hence $x \in N(T)$
So, N(T) is closed.

**14. A continuous function $f: X \to Y$ of topologial spaces preserves a property P if f(A) satisfies P whenever, A satisfies. Which of the following are not preserved by f?**
**(i) Compactness**
**(ii) Connectedness**
**(iii) Openness**
**संस्थिति समष्टि का एक सतत फलन $f: X \to Y$ गुण को संरक्षित करता है, यदि f(A),P को संतुष्ट करता है जब कभी A, P को संतुष्ट करता है। निम्नलिखित में से कौन से के द्वारा संरक्षित नहीं किये जाते हैं?**
**(i) संहता**
**(ii) सम्बद्धता**
**(iii) विवृतता**

(a) (i) (b) (ii)
(c) (i), (ii) (d) (iii)

**Ans. (d) :**Consider a continuous function $f: X \to Y$ defined by $f(x)=x^2$ and A is the open interval (–1, 1) then f(A) is the interval [0,1) which is not open.

Connectedness- The image of a connected space under a continuous map is connected.
PROOF. Let $f: X \to Y$ be continuous map; Let X be connected. Consider the case of a continuous subjective map (by restricting its range to the space $Z \subseteq Y$)
Suppose $Z = A \cup B$ is a separation of Z into two disjoint nonempty sets open in Z. Then $g^{-1}$ (A) and $g^{-1}$ (B) are disjoint sets whose union is X; they are open in X because g is continuous, and non-empty because g is subjective. Therefore, they form a separation of X, contradicting the assumption that X is connected.

Compactness- The image of a compact space under a continuous map is compact.
Proof- Let $f: X \to Y$ be continuous; Let X be compact. Let $\Omega$ be a covering of the set f(X) by sets open in Y. The collection

$$\{f^{-1}(A) | A \in \Omega\}$$

is a collection of sets covering X; these sets are open in X because f is continuous. Hence, finitely many of them, say

$$f^{-1}(A_1), \ldots\ldots, f^{-1}(A_n)$$

cover X. Then the sets $A_1$, ......, $A_n$ cover f(X)

**15. Let X be an infinite set with discrete topology, then which of the following is true?**
**मान लो विविक्त संस्थिति के साथ में एक अनंत समुच्चय है, तो निम्नलिखित में से कौन सा कथन सत्य है?**

(a) X is connected/ X सम्बद्ध है
(b) X is compact/ X संहत है
(c) Every function from X to another topological space is continuous/प्रत्येक फलन X से अन्य संस्थिति समष्टि पर, सतत है।
(d) X is not a $T_1$-space/ X, me$T_1$-समष्टि नहीं है।

**Ans. (c) :** Let X be an infinite set; the collection of all subsets of X is a topology on X, called the discrete topology. Every function from X with discrete topology is continuous because any subset of X is open, so the pre image of every subset of the co-domain is open.

**16. Let X be countably infinite discrete topological space which is homeomorphic to the subspace y of □ with usual topology. Then which of the following can be Y?**
**(i) Q**
**(ii) Z**
**(iii) $\left\{\frac{1}{n}; n \in N\right\} \cup 0$**
**(iv) N**
**मान लो X एक गणनीय अनंत विविक्त संस्थिति है, जो कि सामान्य संस्थिति के साथ के □ उपसमुच्च्य Y से होमियोमोर्फिक है। निम्नलिखित में से कौन सा Y हो सकता है?**

(a) (ii), (iii), (iv) (b) (i), (ii), (iii)
(c) (i), (iii), (iv) (d) (ii), (iv)

**Ans. () :** A homeomorphism is a bijection $f: X \to Y$such that f: $X \to Y$ and $f^{-1}$ : $Y \to X$ are both continouous. Spaces X and Y are homeomprphic.
Now if space X is countably infinite topological space homeomorphic to the subspace Y of R with usual topology then Y can be any one of □ ,□ or□ ; countable sets of R.
Now Let X= $\{0\} \cup \{1/n : n \in N\}$ as a subspace of R and consider N with the discrete - topology (inherited from R). Consider the bijection $f: N \to X$ defined by f(0)=0 and f(n) = 1/n. f is continuous. The inverse bisection is not continuous. {0} is open, but the image f({0}) = {0} is not open in X. This is because any open ball around 0 in R contains some 1/n.

**17. Let X be an uncountable set. Then which of the following collections T of subsets of X is not a topology?**
**मान जो X एक अगणनीय समुच्चय है। तो निम्नलिखित में से कौन-सा X के उपसमुच्चयों का संग्रह T एक संस्थिति नहीं है।**

(a) T = P(X), then power set of X
(b) T = {X,φ}
(c) T = (A ⊆X | A=X or X–A is countable}
(d) $T=\{A\subseteq X|X-A \text{ is countable}\}$

**Ans. (c) :** If X is any set, the collection of all subsets of X is a topology on X, called the discrete topology.
The collection of X and ϕ only is also a topology on X, called the indiscrete topology, or the trivial topology.
Let Ω be the collection of all subsets A of X such that X–A either is countable or is all of X. Then Ω is a topology on X.
Now $T=\{A\subseteq X: A \text{ is finite or } A=X\}$ is not a topology, even though $\phi, X\in\Omega$, any infinite proper subset $Y\subseteq X$ can be written as the union
$y=\bigcup_{Y\in Y}\{y\}$
Each {y} is in Ω, but their union Y is not contained in Ω.

**18. What does the tychonoff Theorem assert?**
**टीचोनोफ्फ (Tychonoff) प्रमेय क्या जोर देता है?**
(a) Product of compact spaces is compact/संहत समष्टिों का गुणन संहत होता है।
(b) Product of connected spaces is connected/सम्बद्ध समष्टिों का गुणन सम्बद्ध होता है।
(c) Product of Hausdorff spaces is Hausdorff/हाउसडॉर्फ समष्टिों का गुणन हाउसडॉर्फ होता है।
(d) Product of normal spaces is normal/सामान्य समष्टियों का गुणन सामान्य होता है।

**Ans. (a) :** Tychonoff Theorem- An arbitrary product of compact spaces in compact in the product topology.

**19. The equation of a normal plane at a point of a space curve $x^i = x^i(s)$ is:**
**आकाश वक्र $x^i = x^i(s)$ के एक बिन्दु पर लम्ब तल समीकरण है:**
(a) $p^i (X^i - x^i)=0$ (b) $t^i(X^i - x^i)=0$
(c) $b^i(X^i - x^i)=0$
(d) None of the above/उपयुक्त में से कोई नहीं

**Ans. (b) :** Let $x^i = x^i$ (s) be a point on the curve and $X^i$ be the position vector of any point on the plane . Then $(X^i-x^i)$ lies in the normal plane since $(X^i-x^i)$ is perpendicular to $\mathbf{t}^i$; tangent vector, we get $\mathbf{t}^i (X^i-x^i) = 0$ as the equation of the normal plane.

**20. A necessary and sufficient condition that a given space curve is a plane curve is that:**
**दिये गये एक आकाश वक्र को समतलीय वक्र होने के लिए आवश्यक और पर्याप्त प्रतिबन्ध है:**
(a) $\tau=0$ (b) K=0
(c) $\tau=K$ (d) $KK'=\tau\tau'$

**Ans. (a) :** Let us take the curve to be a plane curve, since the curve lies in a plane, the osculating plane at every point of the curve is the plane containing the curve itself, so that the binormal **b** is constant.
So $\frac{d\mathbf{b}}{dx}=0\Rightarrow\left|\frac{d\mathbf{b}}{ds}\right|=0.$ Hence $\tau=\left|\frac{d\mathbf{b}}{ds}\right|=0$ at all points of the curve.
Conversely let $\tau=0$ at all points of the curve.
Now $\tau=\left|\frac{d\mathbf{b}}{ds}\right|=0\Rightarrow\mathbf{b}$ is a constant vector and so for any vector R,
$\frac{d}{ds}(\mathbf{r}.\mathbf{b})=\frac{d\mathbf{r}}{ds}.\mathbf{b}+\mathbf{r}.\frac{d\mathbf{b}}{ds}=t.\mathbf{b}+\mathbf{r}.\mathbf{b}'$
∵ **t.b.** = 0 and **b**'= 0 we have $\frac{d}{ds}(\mathbf{r}.\mathbf{b})=0$ for any point **r** on the curve.
Hence $\mathbf{r}\cdot\mathbf{b}$ = constant = c (say). If **r = (x(s), z(s))** and **b** = $(b_1, b_2, b_3)$, then $\mathbf{r}\cdot\mathbf{b}$ = c gives $xb_1 + yb_2 + zb_3 = c$ which shows that **r**(s) = (x(s), y(s), z(s)) lies on the plane $b_1X + b_2Y + b_3Z = c$.
A necessary and sufficient condition that a given curve be a plane curve is that torsion of the curve vanishes i.e. $\tau=0$ at all points of the curve.

**21. If τ is torsion of a curve $x^i = x^i$ (s) in space at a point and $t^i$, $p^i$, $b^i$ are fundamental unit vectors, then τ is defined as:**
**यदि आकाश वक्र $x^i = x^i(s)$ के किसी बिन्दु पर मरोड़ τ है तथा $t^i$, $p^i$, $b^i$ मौलिक इकाई सदिश हो, तो τ परिभाषित है :**
(a) $\left|\frac{dt^i}{ds}\right|$ (b) $\left|\frac{dp^i}{ds}\right|$
(c) $\left|\frac{db^i}{ds}\right|$
(d) None of the above/उपर्युक्त में से कोई नहीं

**Ans. (c) :** The arc-rate of rotation of the osculating plane is expressed by $b^{i'}=\frac{db^i}{ds}$ whose magnitude is the torsion τ.
So, $\tau=\left|\frac{db^i}{ds}\right|$

**22. The equation of tangent plane at a point $\left(x_0^i\right)$ on the surface $F(x^i)=0, i=1,2,3,$ is:**
**सतह $F(x^i)=0, i=1,2,3,$ के एक बिन्दु $\left(x_0^i\right)$ पर स्पर्श तक का समीकरण है:**
(a) $(x^i - x_0^i)t^i = 0$ (b) $(xi - x_0^i)p^i = 0$
(c) $(x^i - x_0^i)b^i = 0$ (d) $(x^i - x_0^i)\frac{\partial F}{\partial x^i} = 0$

**Ans. (d) :** The equation of the tangent plane to a surface $F(x^i) = 0$ at a point $\left(x_0^i\right)$, where i = 1, 2, 3 is derived from the face that the gradient vector ∇F at $\left(x_0^i\right)$ is perpendicular to the tangent plane.

The equation of the tangent plane is:

$$\frac{\partial F}{\partial x^i}\left(x^i - x_0^i\right) = 0,$$

Which corresponds to option (d):

$$\left(x^i - x_0^i\right)\frac{\partial F}{\partial x^i} = 0$$

**23. If $k_1$ and $k_2$ are principal curvatures at a point on the surface, the Gaussian curvature K at that point is:**

**यदि सतह के किसी बिन्दु की मुख्य व वक्रताएँ $k_1$और $k_2$ हों तो सतह के उस बिन्दु की गॉसीय वक्रता है:**

(a) $2(k_1 + k_2)$ (b) $\frac{1}{2}(k_1 + k_2)$

(c) $\sqrt{k_1^2 + k_2^2}$ (d) $k_1k_2$

**Ans. (d) :** If $k_1$ and $k_2$ are principal curvature, at a point on the surface, the Gaussian curvature K is defined as

$K = k_1k_2$.

**24. The principal radii of curvature of the surface:**

$$\mathbf{x\cos\frac{z}{a} = y\sin\frac{z}{a}}$$

**are equal to:**

**सतह**

$$\mathbf{x\cos\frac{z}{a} = y\sin\frac{z}{a}}$$

**की वक्रता की मुख्य त्रिज्याएँ बराबर है:**

(a) $\pm\frac{x^2 + y^2 + a^2}{a}$ (b) $\pm\frac{x^2 + y^2 - a^2}{a}$

(c) $\pm\frac{x^2 - y^2 + a^2}{a}$

(d) None of the above/उपर्युक्त में से कोई नही

**Ans. (a) :** Let us first find the parametric representation of the given surface.

$$\frac{x}{\sin(z/a)} = \frac{y}{\cos(z/a)} = u(\text{say})$$

Further ; $\frac{x}{y} = \tan(z/a)$ so that $(z/a) = \tan^{-1}(x/y)$

Taking, $v = \tan^{-1}(x/y)$ we get $v = z/a \Rightarrow z = av$

Using u and v as parameters, the given surface has the representation

$x = u\sin v,\ y = u\cos v,\ z = av$

Hence the position vector, r of a point P on the curve is

$r = (u\sin v, u\cos v, av)$

$\therefore r_1 = (\sin v, \cos v, 0)$ and $\mathbf{r_2} = (u\cos v, -u\sin v, a)$

Þ $r_1 \times r_2 = (a\cos v, -a\sin v, -u)$

Þ $E = r_1r_1 = 1, F = r_1r_1 = 0, G = r_2r_2 = u^2 + a^2$

Þ $H^2 = EG - F^2 = u^2 + a^2$ and hence H ¹ 0

Þ $r_{11} = (0,0,0), r_{11} = (\cos v, -\sin v, 0)$

$r_{22} = (-u\sin v, -u\cos v, 0), r_{21} = (\cos v, -\sin v, 0)$.

Þ $HL = r_{11}.(r_1 \times r_2) = 0$

$HM = r_{12}(r_1 \times r_2) = a$

$HN = r_{22}(r_1 \times r_2) = -au\sin v\cos v + au\sin v\cos v + 0 = 0$

Since $H \neq 0,\ L = 0,\ N = 0,\ m = a/H = \frac{a}{\sqrt{u^2 + a^2}}$

The principal curvatures are given by

$(EG - F^2)K^2 - (EN$Þ$FM + GL)K + LN - M^2 = 0$

$\Rightarrow (u^2 + a^2)K^2 - \frac{a^2}{\left(\sqrt{u^2 + a^2}\right)^2} = 0$ (on substituting values)

$\Rightarrow K = \pm\frac{a}{(x^2 + y^2 + a^2)}$

$\Rightarrow$ The principal radii are $\pm\frac{x^2 + y^2 + a^2}{a}$

**25. Let z=z(x,y) be a solution of $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1$ passing through (0,0,0) Then z(0,1) is:**

**माना कि z=z(x,y), $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1$ का (0,0,0) से गुजराता हुआ एक हल है। तो z(0,1) है:**

(a) 0 (b) 1

(c) 2 (d) 3

**Ans. (b) :** Given $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1$

Let the complete solution be

$z = ax + by + c$

$\Rightarrow \frac{\partial z}{\partial x} = a\ \&\ \frac{\partial z}{\partial y} = b$ and so $b = 1/a$

$\therefore z = ax + y/a + c.$

Now z = ax + y/a + c passes through origin hence c=0.

Thus z = ax + y/a is the required solution.

$\therefore z(0,1) = 1/a$

**26. Let f(x, y) be a homogeneous polynomial of degree n. Then:**

$$\mathbf{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} =}$$

**माना कि f(x,y) समघातीय बहुपद है। तो:**

$$\mathbf{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} =}$$

(a) f (b) 2f

(c) nf (d) (n–1)f

**Ans. (c) :** If f (x, y) be a homogeneous polynomial of degree n. Then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof We have $f = x^n g(y/x)$

$$\therefore \frac{\partial f}{\partial x} = n x^{n-1} g(y/x) + x^n g'(y/x).\frac{-y}{x^2}$$

$= n\,x^{n-1}g(y/x) - y\,x^{n-2}\,g'(y/x)$

& $\frac{\partial f}{\partial y} = x^n g'(y/x).\frac{1}{x} = x^{n-1}g'(y/x)$

Thus,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n g\left(\frac{y}{x}\right) = nf.$$

**27. Plane curve of fixed perimeter and maximum area is:**
**निर्धारित परिमित वाले समतल वक्र का क्षेत्रफल अधिकतम होगा यदि वक्र है:**

(a) Circle/वृत्त
(b) Hyperbola/अतिपरवलय
(c) Rectangular hyperbola /आयताकार अतिपरवलय
(d) Ellipse/दीर्घवृत्त

**Ans. (a) :** The perimeter and the area under the curve are given by

Perimeter = Arc length = $\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$

Area under the curve A = $\int_{x_1}^{x_2} y(x)\,dx.$

Using Lagrange multiplier Technique, define a new functional F = $y+\lambda\sqrt{1+y'^2}$ and to optimize $\int_{x_1}^{x_2} y+\lambda\sqrt{1+y'^2}\,dx$ we have

$$\frac{\partial F}{\partial y} = 1,\ \frac{\partial F}{\partial y'} = \frac{\lambda y'}{\sqrt{1+y'^2}}$$

Euler's Equation:

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 1$$

$\Rightarrow \quad \frac{\lambda y'}{\sqrt{1+y'^2}} = x + a$

$\Rightarrow \quad \frac{\lambda y'}{x+a} = \sqrt{1+y'^2}$

$\Rightarrow \quad \lambda^2 y'^2 = (1+y'^2)(x+a)^2$

$\Rightarrow \quad y'^2(\lambda^2 - (x+a)^2) = (x+a)^2$

$\Rightarrow \quad y' = \frac{x+a}{\sqrt{\lambda^2 - (x+a)^2}}$

$\Rightarrow \quad y = -\sqrt{\lambda^2 - (x-a)^2} + b$

$\Rightarrow \quad (y-b)^2 = \lambda^2 - (x+a)^2$

$\Rightarrow \quad (x+a)^2 + (y-b)^2 = \lambda^2$

Which is a circle.

**28. Minimum distance between the circle $x^2 + y^2 = 1$ and the straight line $x + y = 4$ is:**
**वृत्त $x^2 + y^2 = 1$ तथा सरल रेखा $x+y = 4$ के बीच की न्यूनतम दूरी है:**

(a) $2\sqrt{2}$ (b) $\frac{1}{2\sqrt{2}}$
(c) $2\sqrt{2}+1$ (d) $2\sqrt{2}-1$

**Ans. (d) :** The minimum distance between the circle $x^2 + y^2 = 1$ and line $x + y = 4$ = (distance of origin from the line $x + y = 4$) – (radius of the circle) $= 2\sqrt{2}-1$

**29. The surface of revolution of a curve $y = y(x)$ is**

$$2\pi\int_{y_1}^{x_2}(y\sqrt{1+y'^2})dx$$

**is minimum when the curve is**
**वक्र का प्रतिवर्तन से प्राप्त सतह:**

$$2\pi\int_{y_1}^{x_2}(y\sqrt{1+y'^2})dx$$

**न्यूनतम होगी यदि वक्र है:**

(a) parabola/परवलय (b) circle/वृत्त
(c) catenary/कैटेनरी (d) ellipse/दीर्घवृत्त

**Ans. (c) :** Here $f = y\sqrt{1+y'^2}$ does not contain x explicitly thus, the Euler's equation reduces to

$f - y'\frac{\partial f}{\partial y'} = c(\text{say})$

$y\sqrt{1+y'^2} - y'\frac{\partial}{\partial y'}.y\sqrt{1+y'^2} = c$

i.e $y\sqrt{1+y'^2} - y'\frac{y}{2}(1+y'^2)^{-1/2}2y' = c$

or $\frac{y}{\sqrt{(1+y'^2)}} = c$

or $y^2 = c^2 + c^2y'^2$

or $y' = \frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$

Separating the variables and integrating, we have

$\int\frac{dy}{\sqrt{y^2-c^2}} = \int\frac{dx}{c} + c'$

$\cosh^{-1}(y/c) = \frac{x+a}{c}$

i.e. $y = c\cos h\left(\frac{x+a}{c}\right)$

Which is centenary.

**30. Fredholm integral equation has a solution:**
**फ्रेडहोम समाकल समीकरण का हल है:**

(a) $\phi(x) = e^x$ (b) $\phi(x) = e^{\log x}$
(c) $\phi(x) = 1$ (d) $\phi(x) = \log x$

**Ans. (d):** The said Fredholm integral equation is missing.

**31. Solution of Volterra integral equation**

**$\varphi(x) = x - \int_0^x (x-t)\varphi(t)dt$**

**With $\varphi_0(x) = 0$, is:**

**वोल्टेरा समाकल समीकरण जबकि का हल है:**

**$\varphi(x) = x - \int_0^x (x-t)\varphi(t)dt$**

**जबकि $\varphi_0(x) = 0$, का हल है:**

(a) $\phi(x) = \cos x$ (b) $\phi(x) = -\sin x$

(c) $\phi(x) = e^x$ (d) $\phi(x) = \sin x$

**Ans. (d) :** Here $\phi(x) = x + \int_0^x (t-x)\phi(t)dt$

On comparing with $y(x)=f(x)+\lambda\int_0^x K(x,t)\phi(t)dt$

We have f(x)=x, $\lambda = 1$, $K(x,t) = (t-x)$

Let $K_m(x,t)$ be the $m^{th}$ iterated kernal. Then

$K_1(x,t)=K(x,t) = (t-x)$

and $K_m(x,t) = \int_t^x K(x,z)K_{m-1}(z,t), m = 2,3, ...$

m = 2 gives

$$K_2(x,t) = \int_t^x K(x,z)K_1(z,t)dz = \int_t^x (z-x)(t-z)dz$$

$$= \left[(t-z)\frac{(z-x)^2}{2}\right]_t^x - \int_t^x (-1)\frac{(z-x)^2}{2}dz$$

$$= 1/2\int_t^x (z-x)^2 dz = 1/2\left[\frac{(z-x)^3}{3}\right]_t^x$$

$$K_2(x,t) = \frac{-(t-x)^3}{3!}$$

Next m = 3 gives

$$K_3(x,t) = \int_t^x K(x,z)K_2(z,t)dz = \int_t^x (z-x)\left\{-\frac{(t-z)^3}{3!}\right\}dz$$

$$= -\frac{1}{3!}\int_t^x (z-x)(t-z)^3 dz = -\frac{1}{3!}$$

$$\left(\left\{(z-x)\frac{(t-z)^4}{(-4)}\right\}_t^x + \int_t^x 1\frac{(t-z)^4}{(-4)}dz\right)$$

$$-\frac{1}{4.3!}\int_t^x (t-z)^4 dz = -\frac{1}{4.3!}\left[\frac{(t-z)^5}{(-5)}\right]_t^x$$

$$\therefore \quad K_3(x,t) = \frac{(t-x)^5}{5!}$$

On observing mathematical induction enables us to write

$$K_m(x,t) = (-1)^{m-1}\frac{(t-x)^{2m-1}}{(2m-1)!}; m = 1,2,3...$$

Now, by the definition of resolvent kernal.

$$R(x,t,\lambda) = \sum_{m=1}^{\infty}\lambda^{m-1}K_m(x,t) = \sum_{m=1}^{\infty}K_m(x,t)$$

$$= K_1(x,t) + K_2(x,t) + K_3(x,t) + ...$$

$$= \frac{t-x}{1!} - \frac{(t-x)^3}{3!} + \frac{(t-x)^5}{5!} + ...$$

$$= \sin(t-x)$$

Finally the required solution is given by

$$y(x) = f(x) + \lambda\int_0^x R(x,t:\lambda)f(t)df = x + \int_0^x t\sin(t-x)dz.$$

$$= x + [t(-\cos(t-x)]_0^x - \int_0^x 1.[-\cos(t-x)df$$

$$= x - x + [\sin(t-x)]_0^x$$

$$\Rightarrow y(x) = \sin x$$

**32. Extremal of $\int_0^1 [y'^2 + 12xy]dx$ with y (0) = 0 and y(1) = 1, is along the curve:**

**y(0) = 0 और y(1) = 1 प्रतिबन्ध के साथ $\int_0^1 [y'^2 + 12xy]dx$ का 'बाह्यतम' ( Extremal ) निम्न वक्र के साथ-साथ ( along ) है:**

(a) $y = x^4$ (b) $y = x^3$

(c) $y = x^2$ (d) $y = x$

**Ans. (b) :** Here $F = y'^2 + 12xy$

Euler Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$12x - 2y'' = 0$ or $y'' = 6x$

$\Rightarrow y' = 3x^2 + C$

$\Rightarrow y = x^3 + Cx + C'$

Applying the conditions y (0) = 0 and y(1) = 1, we get C = C' = 0

Thus, $y = x^3$ is the required curve

**33. Volterra integral equation has a solution:**

**वोल्टेरा समाकल समीकरण का हल है:**

(a) $\phi(x) = (1+x^2)^{-3/2}$ (b) $\phi(x) = (1-x^2)^{-3/2}$

(c) $\phi(x) = (1+x^2)^{-1/2}$ (d) $\phi(x) = (1-x^2)^{-1/2}$

**Ans. ():** The said Volterra integral equation is missing.

**34. What are the coordinates of the reflection of the point (1, 2, 3) in x, y, z space in a mirror along x, z-plane?**

**x,y,z दिकस्थान में एक बिन्दु (1,2,3) का x, z-तल में पड़े एक दर्पण पर बिम्ब के निर्देशांक क्या होंगे?**

(a) (–1, –2, –3) (b) (1, –2, –3)

(c) (1, –2, 3) (d) (–1, 2, 3)

**Ans. (c) :** Reflection r of (x, y, z) about xz-plane is

$$r(x, y, z) = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

∴ The point (1, 2, 3) after reflection along xz-plane is given by

$$\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}1\\-2\\3\end{bmatrix}$$

**35. Consider two waves of same angular frequency ω, same angular wave number k, same amplitude a travelling in the positive direction of x-axis with the same speed and with phase difference $\phi$. Then the superposition principle yields a resultant wave with:**

**एक ही कोणीय बारंबाराता ω, एक ही कोणीय तरंग संख्या k, एक ही आयाम a के दो तरंगों पर विचार कीजिए जो एक ही गति कलांतर $\phi$ के साथ x-अक्ष के धन दिशा में प्रगामी है। तो अध्यारोपण नियम एक परिणामी तरंग देता है इनके साथः**

(a) Amplitude 2a and phase $\phi$/आयाम 2a तथा कला $\phi$

(b) Amplitude 2a and phase ($\phi$/2)/आयाम 2a तथा कला ($\phi$/2)

(c) Amplitude 2a cos($\phi$/2)and phase ($\phi$/2)/आयाम 2a cos($\phi$/2) तथा कला ($\phi$/2)

(d) Amplitude 2a cos ($\phi$/2) and phase $\phi$/आयाम 2a cos ($\phi$/2) तथा कला $\phi$

**Ans. (c) :** Let the two sinusoidal waves with given conditions be

$y_1 = a\sin\omega t,\quad y_2 = a\sin(\omega t+\phi)$

Principle of superposition dictates that the resultant wave will be given by

$y = y_1 + y_2 = a\sin\omega t + a\sin(\omega t+\phi)$

$= a[\sin\omega t + \sin(\omega t+\phi)]$

$= a\left[2\sin\left(\omega t+\frac{\phi}{2}\right)\cos\frac{\phi}{2}\right]$

$= \left(2a\cos\frac{\phi}{2}\right).\sin(\omega t+\frac{\phi}{2})$

Hence, the resultant wave is of amplitude

$2a\cos\left(\phi/2\right)$ and phase difference $\frac{\phi}{2}$

**36. Pick all the correct options in the following problem:**

**Consider a particle of mass m in simple harmonic oscillation about the origin with spring constant k, then for the Lagrangian L and Hamiltonian H of the system:**

**निम्नलिखित समस्या में सत्य विकल्पों का चुनाव कीजिएः**

**निर्देश मूल बिन्दु के आरपार कमानी स्थिरांक k तथा द्रव्यमान m युक्त सरल आवर्त दोलन करते हुए एक कण पर विचार कीजिए। तो निकाय के लैग्रांजी L तथा हैमिल्टनी H के लिएः**

(i) $L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$H(x,\dot{x}) = \frac{p^2}{2m} + \frac{1}{2}kx^2$, p is generalized momentum/व्यापकीकृत संवेग है

(ii) $L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ and the generalized momentum is p= m $\dot{x}$ /तथा व्यापकीकृत संवेग है

(iii) $L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ and the generalized momentum is p = m $\dot{x}$ /तथा व्यापकीकृत संवेग है

(iv) $L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$,

(a) (i), (iii), (iv) (b) (i), (ii), (iv)
(c) (i), (iii) (d) (ii), (iii), (iv)

**Ans. (c) :** Lagrangian $L \equiv T - V$

where T and V are kinetic and potential energies.

Lagrangian L of the described system of mass m being at the end of spring in SHM with spring constant k is

$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

Hamiltonian H of the described system is

$H = p\dot{x} - L : p - \text{conjugate momentum} = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$= p\dot{x} - \left(\frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2\right)$

$= p\left(\frac{p}{m}\right) - \frac{m}{2}\left(\frac{p}{m}\right)^2 + \frac{k}{2}x^2$

$= \frac{p^2}{2m} + \frac{kx^2}{2}$

where we now have H in terms of x and p, with n $\dot{x}$'s.
H is simply the energy, expressed in terms of x and p.

**37. The Euler's equation of motion for the perfect fluid, with $\vec{F}$ as the external force per unit mass of the fluid, is:**

**यदि किसी तरल पदार्थ पर प्रति इकाई परिमाण पर लगने वाला बाह्य बल $\vec{F}$ हो तो पूर्ण तरल पदार्थ का आयलर का गति समीकरण हैः**

(a) $\frac{D\vec{q}}{Dt} = \vec{F} + \frac{1}{\rho}\nabla p$ (b) $\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p$

(c) $\frac{D\vec{q}}{Dt} = \rho\vec{F} - \nabla p$ (d) $\rho\frac{D\vec{q}}{Dt} = \vec{F} + \nabla p$

**Ans. (b) :** The Euler's equation of motion of an in viscid (perfect)fluid

$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p$

where $\vec{F}$ is the external force per unit mass of the fluid, p and ρ are the pressure and density respectively.

**38. The stream function $\psi$, for a two-dimensional motion, is:**
**द्विआयामी गति के लिए प्रवाहदिशि फलन $\psi$ है:**
(a) zero along a stream line/प्रवाह रेखा की दिशा में शून्य
(b) variable along a stream line/प्रवाह रेखा की दिशा में चर
(c) constant along a stream line/प्रवाह रेखा की दिशा में अचर
(d) None of the above/उपर्युक्त में से कोई नहीं

**Ans. (c) :** For a two dimensional incompressible flow parallel to the xy-plane in a rectangular Cartesian coordinate system, let u and v denote the velocity vectors in the x and y direction respectively.

Equation of continuity: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

A function $\psi(x,y,t)$ defined such that

$u=\frac{\partial \phi}{\partial y}$ and $v=-\frac{\partial \varphi}{\partial x}$

which gives that $\psi$ satisfies the equation of continuity $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ is known as stream function.

Since $\psi$ is a point function then

$d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=-vdx+udy$

Now, the equation of stream line

$\frac{u}{dx}=\frac{v}{dy}$ or $udy-vdx=0$ gives that $d\psi=0$ on a stream line.

$\Rightarrow \psi$ is constant along a streamline.

**39. Let G be a multiplicative group of positive real numbers and G' be the additive group of real numbers. Then the mapping**

$$f:G\to G'$$

**Given by**

$$f(x)=\log x, \forall x\in G,$$

**is:**
**मान लिया कि धनात्मक वास्वतिक संख्याओं का गुणनीय समूह है तथा वास्वतिक संख्याओं का योजनीय समूह है। प्रतिचित्रण**
**$f:G\to G'$ जो कि दिया गया है:**

$$f(x)=\log x, \forall x\in G,$$

**है:**
(a) one-one, onto but not homomorphism/एकैकी, आच्छादक परन्तु समरूपता नहीं
(b) one-one, homomorphism but not onto/एकैकी, समरूपता परन्तु आच्छादक नहीं
(c) onto, homomorphism, but not one-one/आच्छादक, समरूपता परन्तु एकैकी नहीं
(d) one-one, onto and homomorphism/एकैकी, आच्छादक और समरूपता

**Ans. (d) :** If $f(x_1)=f(x_2)$ for some $x_1, x_2\in G$ then $\log_e x_1=\log_e x_2$, gives that $x_1=x_2$, proving the injectivity of f(x).
Let $y\in G$ then for some $x\in G$ we have $x=e^y$ which gives $\log_e(x)=\log_e(e^y)=y$, proving surjectivity of f.
Now, we have $f(x_1x_2)=\log_e(x_1.x_2)$
$=\log_e(x_1)+\log_e(x_2)$
$=f(x_1)+f(x_2)$
Showing that f is homomorphism.

**40. Which of the following statements is not correct?**
**निम्नलिखित में से कौन सा कथन सत्य नहीं हैं?**
(a) Product of two odd permutations is an odd permutation/दो विषम क्रमचयों (permutations) का उत्पाद एक विषम क्रमचय है।
(b) Product of an even permutation and an odd permutation is an odd permutation/एक सम क्रमचय (permutations) और एक विषम क्रमचय का उत्पाद (product) विषम क्रमचय है।
(c) Identity permutation is an even permutation/तत्समक क्रमचय (identity permutations) एक सम क्रमचय है।
(d) Every group has at least two normal subgroups/प्रत्येक समूह में कम से कम दो प्रसामान्य उपसमूह हैं।

**Ans. (a) :**
a) Product of two even or odd permutations is even permutation.
b) Product of an even permutation and an odd permutation is odd permutation
c) Identity permutation for the finite n-element set $\{1,2,...n\}$ given by $e=\begin{pmatrix}1 & 2 & ...n\\1 & 2 & ...n\end{pmatrix}$ is an even permutation because it can be written as product of even number of transpositions
d) Every group has at least two normal subgroups i.e. the trivial subgroups, {0} and group itself.

**41. For the multiplicative group of residue classes $\{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6}(\text{mod}7)\}$ the generating element is:**
**अवशेष वर्गों $\{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6}(\text{mod}7)\}$ के गुणनात्मक समूह का जनक अवयव है:**
(a) $\bar{2}$ (b) $\bar{3}$
(c) $\bar{4}$ (d) $\bar{6}$

**Ans. (b) :** $<\bar{1}>=\{\bar{1}\}$
$<\bar{2}>=\{\bar{1},\bar{2},\bar{4}\}$
$<\bar{3}>=\{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6}\}=U_7$ (Multiplicative group modulo7)
$<\bar{4}>=\{\bar{1},\bar{2},\bar{4}\}$ & $<\bar{6}>=\{\bar{1},\bar{6}\}$

**42. If p is a prime number and G is a non-Abelian group of order $p^3$, then the number of elements in the centre of G is exactly:**
**यदि एक रूढ़ संख्या हे और एक अन-अबेलीय क्रम का समूह है, तो के केन्द्र में अवयवों की यथार्थ संख्या है:**

(a) $p-1$ (b) $p$
(c) $p+2$ (d) $p^2$

**Ans. (b) :** Let G be a group of order $p^3$; p-prime number.
Since G is a p-group then its center Z is not the trivial group. So the order Z must be p or $p^2$ or $p^3$.
If the order of Z is $p^3$, then Z = G and which gives that G is abelian, and G is not abelian. So order of $Z \neq p^3$.
Suppose order of $Z = p^2$ then $|G/Z| = p$; a cyclic group and that implies G is abelian, and G is not abelian. So order of $Z \neq p^2$. Thus we get $|z| = p$

**43. Which statement is correct?**
**निम्नलिखित में कौन सा कथन सत्य है?**

(a) The set of integers I is a subring as well as an ideal of the ring of rational numbers (Q,+,.)./पूर्णांकों का समुच्चय I परिमेय संख्याओं के वलय (Q, +,.) का उपवलय तथा गुणजावली (ideal) है।
(b) The set of integers I is only a subring but not ideal of the ring of rational numbers (Q,+,.) /पूर्णांकों का समुच्चय I, परिमेय संख्याओं के वलय (Q, +,.) का मात्र उपवलय है परन्तु गुणजावली नहीं है।
(c) The set of integers 1 is neither a subring nor an ideal of the ring of rational numbers (Q,+,.)./पूर्णांकों का समुच्चय I परिमेय संख्याओं के वलय (Q, +,.) का गुणजावली है, परन्तु उपवलय नहीं है।
(d) The set of integers I is an ideal but not a subring of the ring of rational numbers (Q,+,.)./पूर्णांकों का समुच्चय, I परिमेय संख्याओं के वलय (Q, +,.) का गुणजावली है, परन्तु उपवलय नहीं है।

**Ans. (b) :** The set of integer I is a subring of ring of rational numbers (Q,+,.) under the usual addition and multiplication of rational numbers.
Now let $3 \in I \text{ and } \frac{1}{2} \in (Q,+,.) \text{ then } 3.\frac{1}{2} \notin I$ and hence I is not an ideal of (Q,+,.).

**44. An ideal S of a commutative ring R with unity is maximal if and only if the residue class ring R| S is:**
**इकाई रखने वाले क्रमविनिमेय वलय R का एक गुणजावली S अधिकतम (maximal) है यदि और केवल यदि अवशेष वर्ग वलय R| S है:**

(a) a commutative ring/क्रमविनिमेय वलय
(b) a commutative ring with unity/इकाई रखने वाला क्रमविनिमेय वलय
(c) an integral domain with unity/इकाई रखने वाला एक पूर्णकीय प्रान्त (domain)
(d) a field/एक क्षेत्र

**Ans. (d) : <u>Proposition</u>-** An ideal S of a commutative ring R with identity is maximal if and only if R/S is a field.
**<u>PROOF</u>-** Since every ideal of R/S is of the form B/S, where B is an ideal containing S, it follows that S is a maximal ideal if and only if R/S is without proper ideals, and a commutative ring with identity is a field if and only if it is without proper ideals.

**45. Which statement is false?**
**कौन सा कथन असत्य है?**

(a) The set R [x] of all polynomials over a ring R is a ring./किसी वलय R पर सभी बहुपदों का समुच्चय R [x] एक वलय है।
(b) The set D [x] of all polynomials over an integral domain D is an integral domain./किसी पूर्णांकीय प्राप्त D पर सभी बहुपदों का समुच्चय D [x] एक पूर्णाकीय प्रान्त है।
(c) The set F [x] of all polynomials over a field F is a field./किसी क्षेत्र F पर सभी बहुपदों का समुच्चय F [x] एक क्षेत्र है।
(d) If D is an integral domain with unity, then any unit in D [x] must already be a unit in D./यदि D एक इकाई रखने वाला पूर्णांकीय प्रान्त हो, तो D [x] में कोई इकाई D में भी पूर्व से ही इकाई होना चाहिए।

**Ans. (c) : <u>Definition</u> -** A polynomial with coefficients in a ring R is a (finite) linear combination of powers of the variable

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ..... + a_1 x + a_0$$

The set of polynomials with coefficients in a ring R will be denoted by R[x].
If $g(x) = b_m x^m + b_{m-1}x^{m-1} + .... + b_1 x + b_0$
is another polynomial with coefficient in R, then f(x) and g(x) are equal if $a_i = b_i$ for i=0,1,2,.....
Now
$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + ..... +$
$= \sum_K (a_k + b_k)x^K; a_i + b_i$ is addition in R
and $f(x)g(x) = (a_0 + a_1 x + ....)(b_0 + b_1 x + ....)$
$= \sum_{i,j} a_i b_j x^{i+j}; a_i b_i$ is product in R.
There is a unique commutative ring structure on the set of polynomials R [x] with respect to the binary operations defined above. The ring R becomes a subbing of R[x] when the elements of R are identified with the constant polynomial.
**<u>Proposition</u>-** If D is an integral domain then D[x] is an integral domain.
**<u>Proof</u>-** We know that D[x] is a ring and so clearly D(x) is commutative whenever D is commutative. If 1 is the unity element of D, f(x)=1is the unity element of D[x]. now suppose that
$f(x) = a_n x^n + a_{n-1}x^{n-1} + ..... + a_1 x + a_0$
and $g(x) = b_m x^m + b_{m-1}x^{m-1} + .... + b_1 x + b_o$
where $a_n \neq 0$ and $b_m \neq 0$ .Then by definition f(x)g(x) has leading coefficient $a_n b_m$ and since, D is an integral domain, $a_n b_m \neq 0$
**<u>Proposition</u>-** If D be an integral domain with unity. Then units of D[x] are those of D.
**<u>Proof</u>-** Since D is a subring of D[x] and the unity of D is that of D[x], units of D are also units of D[x]. Let f(x) be a unit of D[x]. Then there exists $g(x) \in D[x]$ such that

f(x).g(x)=1. Comparing the degrees of both the sides, we have that deg(f(x)) =0=deg(g(x)). Hence f(x) and g(x) both belong to D. This shows that units of D[x] are also units of D.

**Proposition-** F[x]; the set of polynomials over a field F can never be a field

**Proof-** $x \neq 0,$ and it cannot be a unit of F[x].

**46. The degree of the field:**

$$Q\left(\sqrt{2},\sqrt{3}\right)\overset{\text{def}}{=}\left\{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3} : a,b,c,d\in Q\right\}$$

**over Q is:**

**Q पर क्षेत्र:**

$$Q\left(\sqrt{2},\sqrt{3}\right)\overset{\text{def}}{=}\left\{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3} : a,b,c,d\in Q\right\}$$

**की कोटि है:**

(a) 1 (b) 2

(c) 3 (d) 4

**Ans. (d) :** Consider the field $Q\left(\sqrt{2},\sqrt{3}\right)$ generated over Q by $\sqrt{2}$ and $\sqrt{3}$. Since $\sqrt{3}$ is of degree 2 over Q, the degree of the extension $Q\left(\sqrt{2},\sqrt{3}\right)/Q\left(\sqrt{2}\right)$ is at most 2 and is precisely 2 if and only if $x^2–3$ is irreducible over $Q\left(\sqrt{2}\right)$. Since this polynomial is of degree 2, it is reducible only if it has a root; i.e. if and only if $\sqrt{3}\in Q\left(\sqrt{2}\right)$.

Suppose $\sqrt{3}=a+b\sqrt{2}$ with $a,b\in Q$ .Squaring this we obtain $3=a^2+2b^2+2ab\sqrt{2}$. If $ab\neq 0$, then we can solve this equation for $\sqrt{2}$ in terms of a and b which implies that $\sqrt{2}$ is rational, and $\sqrt{2}$ is not rational. If b = 0, then we would have that $\sqrt{3}$ = a is rational and $\sqrt{3}$ is not rational. Finally if a = 0 we have $\sqrt{3}=b\sqrt{2}$ and multiplying both sides by $\sqrt{2}$ we see that $\sqrt{6}$ would be rational and $\sqrt{6}$ is not rational. These shows $\sqrt{3}\notin\left(Q\sqrt{2}\right)$, proving $[Q\left(\sqrt{2},\sqrt{3}\right):Q]=4$

**47. Let f(x)=0 be an equation. The sufficient condition for the convergence of Newton Raphson's iteration method for finding a real root of the above equation is:**

**मान लिया कि f(x) = 0 एक समीकरण है। उपर्युक्त समीकरण के एक वास्तविक हल प्राप्त करने की न्यूटन-रैफसर की दुहराव विधि के संसृत (convergent) होने का पर्याप्त प्रतिबन्ध है:**

(a) $|f(x)f''(x)|=[f'(x)]^2$ (b) $|f(x)f''(x)|<[f'(x)]^2$

(c) $|f(x)f''(x)|>[f'(x)]^2$ (d) $[f(x)f''(x)]=|f'(x)|$

**Ans. (b) :** Newton-Raphson formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \qquad ...(i)$$

On comparing with (i) with

$x_{n+1}=\phi(x)$ : iterative method

we have $\phi(x)=x-\dfrac{f(x)}{f'(x)}$

$$\phi(x)=1-\frac{[f'(x)f'(x)-f(x)f''(x)]}{[f'(x)]^2}$$

Which gives $\phi'(x)=\dfrac{f(x)f''(x)}{[f'(x)]^2}$

Which converges if

$|\phi'(x)|<1$

$\Rightarrow |f(x).f''(x)|<[f'(x)]^2$

for newton-Raphson formula

**48. Given the initial value problem $y'=\dfrac{dx}{dy}=f(x,y)$, where $y(x_0)=y_0$.In Runge-Kutta method:**

**दिये गये प्रारम्भिक मान समस्या $y'=\dfrac{dx}{dy}=f(x,y)$, जहाँ $y(x_0)=y_0$ है। रूंगे-कट्टा विधि में:**

(a) $k_1 = hf(x_n)$ (b) $k_1 = hf(x_n,y_n)$

(c) $k_1 = f(y_n)$ (d) $k_1 = f_1(x_n)–h$

**Ans. (b) :** The Runge-Kutta method computes approximate values $y_1$, $y_2$, ..... $y_n$ of the solution of an initial value problem

$y'=\dfrac{dy}{dx}=f(x, y)$ , $y(x_0) = y_0$

The correct expression for $k_1$ in the Runge-kutta method depends on the specific from f(x, y) generally in the first stage of the method

$k_1 = h\ f(x_n, y_n)$

Where h is size

$x_n$ is the current value of x, $y_n$ is the current value of y

Therefore correct option is b.

**49. Let V be a vector space over F of dimension n. Then which of the following statements is not true?**

**मान लीजिए V एक विमीय सदिश समष्टि F पर है। निम्नलिखित में से कौन सा कथन सत्य नहीं हैं?**

(a) Any set of n vectors is a basis./ n सदिश का कोई भी समुच्चय एक आधार है।

(b) A set of n + 1 vectors is linearly dependent./ n + 1 सदिश का समुच्चय रैखिक निर्भर है।

(c) A set of n vectors which spans V is a basis./ n सदिश का समुच्चय जो कि V का विस्तार करता है, एक आधार है।

(d) A set of n linearly independent vectors is a basis/n रैखिक स्वतंत्र सदिश का समुच्चय एक आधार है।

**Ans. (a) :** Let V be a vector space of dimension n over the field F. The dimension n implies that any basis of V consists of exactly n linearly independent vectors.

Now, let's analyze each statement:

(a) Any set of n vectors is a basis
This is not true. A set of n vectors in V is a basis if and only if the vectors are both linearly independent and span V. Merely having n vectors does not guarantee these properties. For example, a set of n vectors that are linearly dependent will not from a basis.
(b) A set of n + 1 vectors is linearly dependent
This is true. In a vector space of dimension n, any set containing more than n vectors is necessarily linearly dependent, as it exceeds the maximum number of linearly independent vectors.
(c) A set of n vectors which spans V is a basis
This is true. If a set of n vectors spans V, it must be linearly independent, as the dimension of V is n, Hence, it satisfies the definition of a basis.
(d) A set of n linearly independent vectors is a basis
This is true. If a set of n vectors is linearly independent in V, it must also span V, as the dimension of V is n. Therefore, it is a basis.

**50. Let V be a finite dimensional real vector space and f and g be non-zero linear transformation of V to ℝ (The set of real numbers). Assume that ker (f) ⊂ ker (g).Which of the following statements are true?**
**मान लीजिए V एक परिमित विमीय वास्तविक सदिश समष्टि है और f और g, V से ℝ (वास्तविक संख्याओं का समुच्चय) तक शून्येतर रैखिक रूपान्तरण है। माना कि ker (f) ⊂ ker (g)। निम्नलिखित में से कौन सा कथन सत्य है?**
(i) ker(f) = ker(g)
(ii) $f=\lambda g$ for some real number $\lambda \neq 0$ / $f=\lambda g$ कुछ वास्तविक संख्या $\lambda \neq 0$ के लिए
(iii) The linear map $A:V\to R^2$ defined by Ax=(f(x),g(x)), is onto/रैखिक प्रतिचित्रण A:V $\to R^2$ जो कि Ax=(f(x),g(x)) से परिभाषित है, एक आच्छादक है।
(a) (i), (ii) (b) (ii), (iii)
(c) (i), (iii) (d) (iii)

**Ans. (a) :** If $f,g:V\to IR$ are non-zero linear functional, then there exists $v_0\in V$ such that $f(v_0)\neq 0$. As
$f(f(v_0)v - f(v)v_0) = f(v_0)f(v) - f(v)f(v_0) = 0$ ; $\forall v \in V$
∴ We conclude that
$g(f(v_0)v - f(v)v_0) = f(v_0)g(v) - f(v)g(v_0) = 0$ ; $\forall v \in V$
By hypothesis ker (f) ⊂ ker (g)
Then; $g(v)=\frac{g(v_o)}{f(v_o)}f(v)$ ; $\forall v\in V$
$\therefore\ f-\frac{f(v_o)}{g(v_o)}g(v)$, proving that $f=\lambda g$ ; for some $\lambda\neq 0$
Thus, we conclude that
ker (f) = ker (g)
as well by above.

**51. Let V be a finite dimensional real vector space and let $A:V\to V$ be a linear map such that $A^2$=A. Assume that $A\neq 0$ and that $A\neq I$, which of the following statements are true?**
**मान लीजिए V एक परिमित विमीय वास्तविक सदिश समष्टि है और मान लीजिए $A:V\to V$ जो कि $A^2=A$ एक रैखिक प्रतिचित्रण है। मान लीजिए $A\neq 0$ और $A\neq I$, निम्नलिखित में से कौन-कौन से कथन सत्य है?**
(i) Ker (A)$\neq$\{0\}
(ii) V = Ker (A) ⊕ Range {A}
(iii) The map I+A is invertible./प्रतिचित्रण I+A व्युत्क्रमणीय है।
(a) (i),(ii) (b) (ii)
(c) (i),(iii) (d) (i),(ii),(iii)

**Ans. (d) :** Let $v\in V$ such that $A(v)\neq v$ (because $A\neq 0$ and $A\neq I$) for some $v\neq 0\in V$.
Then $A(I-A)(v) = 0$
Which gives $(I-A)v \neq 0 \in \ker(A)$
Thus ker (A) $\neq \{0\}$.
Now, $V=Av+(I-A)v$ : $\forall v\in V$ (Av is Range(A))
$\Rightarrow V=\ker(A)\oplus \text{Range}(A)$
Suppose (I+A)v=0
$\Rightarrow Av=-v\Rightarrow A^2v=-Av\Rightarrow Av=$
$-Av\Rightarrow Av=0\Rightarrow v=0$
Thus, (I+A) is invertible because ker (I+A) is trivial.

**52. Let $x\in ℝ^2$ be a non-zero column vector and A = $xx^T$. Then what is the rank of A?**
**मान लीजिए $x\in ℝ^2$ शून्येत्तर कॉलम सदिश है और A = $xx^T$, तो की कोटि (rank) क्या है?**
(a) 2 (b) 1
(c) n (d) 0

**Ans. (b) :** Let
$x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ then $x^T=\begin{bmatrix}x_1\\x_2\end{bmatrix}^T=[x_1x_2]$
$\therefore xx^T=\begin{bmatrix}x_1\\x_2\end{bmatrix}[x_1\ x_2]=\begin{bmatrix}x_1^2 & x_1x_2\\x_1x_2 & x_2^2\end{bmatrix}$
$=\begin{bmatrix}x_1 & x_2\\0 & 0\end{bmatrix}\left\{R_2\to R_2-\frac{x_2}{x_1}R_1\right\}$
∴ Rank of A = $(xx^T)$ is 1.

**53. Let A and B be square matrices with complex entries. Which of the following conditions assure the similarity of A and B?**
**मान लीजिए A और B वर्ग आव्यूह हैं, जिसकी प्रविष्टियाँ सम्मिश्र संख्याएँ है। निम्नलिखित में से कौन सी अवस्था A और B की समानता सुनिश्चित करती है?**
(i) A and B have the same Eigen values/ A और B का समान eigen मूल्य हो
(ii) A and B are diagonalizable/ A और B विकर्णीय हों
(iii) A and B have the same Jordan form/ A और B का समान Jordan प्रारूप हो
(iv) A and B represent a linear transformation of a vector space with respect to different bases./A और B अलग-अलग आधार के सदिश समष्टि का रैखिक प्रतिचित्रण निरूपित करते है।

(a) (i) (b) (ii)
(c) (iii) and (iv) (d) (i) and (ii)

**Ans. (c) :** To determine which conditions assure that A and B are similar, let's analyze the options:
1. Condition (i): A and B have the same eigenvalues.
Similar matrices always have the same eigenvalues, but the converse is not necessarily true. For instance, two matrices with the same eigenvalues might not be similar if their Jordan forms differ.
This condition alone does not assure similarity.
2. Condition (ii): A and B are diagonalizable.
Both being diagonalizable doesn't imply similarity unless they are diagonalizable to the some diagonal form. Hence, this condition alone is insufficient.
3. Condition (iii): A and B have the same Jordan form.
If two matrices have the same Jordan form, they are necessarily similar. This condition assures similarity.
4. Condition (iv): A and B represent the same linear transformation with respect to different base.
If two matrices represent the same linear transformation in different bases, they are similar by definition.

**54. Let A be a 4 × 4 invertible martix with real entries. Which of the following statements is not necesarily true?**
**मान लीजिए A एक 4 × 4 व्युत्क्रमणीय आव्यूह है जिसकी प्रविष्टियाँ वास्तविक संख्याएँ हैं। निम्नलिखित में से कौन सा कथन सत्य होना आवश्यक नहीं है?**

(i) The rows of A form a basis of $R^4$/ A की पक्तियाँ $R^4$की आधार बनाती हैं
(ii) Null space of A contains only the zero vector/A की शून्य समष्टि में केवल शून्य सदिश शामिल होते हैं।
(iii) A has 4 distinct eigen values/A के 4 भिन्न eigen मूल्य हैं

(a) (i) (b) (ii)
(c) (iii) (d) (i) and (ii)

**Ans. (c) :** Consider the identity matrix

$$I=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}_{4\times4}$$

Now I is invertible then null space of I contains only zero vector thus rank of I is 4 which implies that the rows of I forms a basis for $R^4$.

But I does not possess distinct Eigen values because characteristic polynomial of I is given by

$(1-\lambda)^4$ which has $\lambda = 1$ as its repeated root .

**55. Let $A_n$ be the matrix whose (i, j)$^{th}$ entry is given by $2\delta_i, j-\delta_{i+1,j}-\delta_{i,j+1}$. Then det $A_n$=:**

(a) n (b) n+1
(c) n–1 (d) n+2

**Ans. (b) :** We are given the matrix $A_n$ whose (i, j)$^{th}$ entry is defined as
$a_{ij} = 2\delta_{ij} - \delta_{i+i,\,j} - \delta_{i,\,j+1}$
where $\delta_{\text{ता}}$ is the kronecker delta function
($\delta_{ij} = 1$ if i = j and $\delta_{ij} = 0$ if $i \neq j$ )
this definition implies that $A_n$ is a tridiagonal matrix with following entries
Then,

$$An=\begin{bmatrix}2&-1&0&-&-&0\\-1&2&-1&-&-&0\\0&-1&2&-&-&0\\-&-&-&-&-&-\\-&-&-&-&-&-1\\0&0&0&-&-&2\end{bmatrix}.$$

$D_n = 2D_{n-1} - D_{n-2}$
Where,
$D_n = \det(A_n)$
$$D_1 = 2,\ D_1 = \det\begin{bmatrix}2&-1\\-1&2\end{bmatrix} = 3$$
$D_3 = 4, D_4 = 5, D_5 = 6$
As so on
For this seem that $D_n = n + 1$

**56. Let V be a finite dimensional linear product space with inner product $\langle,\rangle$. For a subset $S \neq \varphi$ of V, let $S^\perp = \{v|\langle v,w\rangle = 0 \forall w \in S\}$. If 0 $0 \notin S$ which of the following is true?**
**मान लीजिए V अंतर गुणन $\langle,\rangle$ के साथ एक परिमित विमीय अंतर गुणन समष्टि है। V के उपसमुच्चय $S \neq \varphi$ के लिए मान लीजिए कि $S^\perp = \{v|\langle v,w\rangle = 0 \forall w \in S\}$ ।यदि $0 \notin S$ तो निम्नलिखित में से कौन सा कथन सत्य है?**

(a) $S = S^\perp$ (b) $S \subset S^{\perp\perp}$
(c) $S^\perp = V$ (d) $S^{\perp\perp} = (0)$

**Ans. (b) :** If $w \in S$ and $v \in S^\perp$, we have <v, w>=0, and <w, v>=0; since v is arbitrary <w, v>=0 for all $v \in S^\perp$ which implies that $w \in S^{\perp\perp}$ and $0 \in S^{\perp\perp}$ but $0 \notin S$
$\therefore S \subset S^{\perp\perp}$

**57. The matrix of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x-2y, y)$ with respect to the standard basis {(1,0),(0,1)} is:**
**मानक आधार {(1,0),(0,1)} के सापेक्ष में रैखिक रूपांतरण $T: \mathbb{R}^2 \to \mathbb{R}^2$ जो कि $T(x,y) = (x-2y, y)$ द्वारा परिभाषित है, का आव्यूह है:**

(a) $\begin{bmatrix}1&0\\1&-3\end{bmatrix}$ (b) $\begin{bmatrix}1&-3\\1&0\end{bmatrix}$
(c) $\begin{bmatrix}1&-3\\1&1\end{bmatrix}$ (d) $\begin{bmatrix}1&-2\\0&1\end{bmatrix}$

**Ans. (d) :** Here T(1,0) = (1,0) = 1(1,0) + 0(0,1)
T(0,1) = (–2,1) = –2(1,0) + 1(0,1)
$$\therefore \text{The matrix} = \begin{bmatrix}1&-2\\0&1\end{bmatrix}$$

**58. What is the nullity of the matrix $\begin{bmatrix} A_n & C \\ 0 & B_m \end{bmatrix}$, if $A_n$ and $B_m$ are invertible matrices of order n and m respectively?**

**आव्यूह $\begin{bmatrix} A_n & C \\ 0 & B_m \end{bmatrix}$, की शून्यता क्या है यदि $A_n$ और $B_m$ क्रमशः n और m कोटि के व्युत्क्रमणीय आव्यूह हैं?**

(a) 0 (b) n
(c) m (d) n + m

**Ans. (a) :** A matrix A of order m × n with rank r can be placed in the form $\begin{bmatrix} I_r & O_{r\times n-r} \\ O_{m-r\times r} & O_{m-r\times n-r} \end{bmatrix}$

Using elementary row and column matrices; where $I_r$ is the r × r identity matrix and $O_{m\times n}$ is the m×n zero matrix.

An invertible matrix has a full rank and nullity is 0.

So, the matrix $\begin{bmatrix} A_n & C \\ 0 & B_m \end{bmatrix}$ has rank n + m and

$$\text{Nullity}\left(\begin{bmatrix} A_n & C \\ 0 & B_m \end{bmatrix}\right) = 0$$

**59. Let f:[0,1] → R be a function defined by f(x) = 10, x ∈ [0,1], then the Lebesgue integral $\int_{Q\cap[0,1]} f d\lambda$, where Q is the set of rational numbers, is given as:**

**मान लिया कि f:[0,1] → R, एक फलन इस प्रकार परिभाषित है कि f(x) = 10, x ∈ [0,1], यदि परिमेय Q संख्याओं का समुच्च्य है, तो लबेग समाकलक $\int_{Q\cap[0,1]} f d\lambda$, है:**

(a) ∞ (b) 1
(c) 10 (d) 0

**Ans. (d) :** Because Q∩[0,1] is a set of measure zero we must have $\int_{Q\cap[0,1]} f d\lambda := 10\times\lambda(Q\cap[0,1]) = 0$

**60. Let α be a bounded function on [a,b], then true statement is:/मान लिया कि α पर एक बन्धित फलन है तो सत्य कथन है:**

(a) α is always a function of bounded variation/α सर्वदा एक बन्धित विभिन्नता वाला फलन है।
(b) α is always a continuous function/α सर्वदा एक सतत् फलन है।
(c) α is always a constant function/α सर्वदा एक अचर फलन है।
(d) sup $\{|\alpha(x)| : x\in[a,b]\}$ always exists/ sup $\{|\alpha(x)| : x\in[a,b]\}$ सर्वदा मूर्तमान होता है।

**Ans. (d) :** If α is a bounded function on [a,b] then Axiom of Completeness dictates that

$$\sup\{|\alpha(x)| : x\in[a,b]\}$$

always exists.

**61. Let λ be the Lebesgue measure on R and Q be the set of all rational numbers. Which of the following is true?**

**मान लिया कि λ, R पर लबेग मापक है तथा Q सभी परिमेय संख्याओं का समुच्चय है। निम्नलिखित में से कौन सा कथन सत्य है?**

(a) $\lambda(Q) = \infty$ (b) $\lambda(Q) = 1$
(c) $\lambda(Q) = 0$ (d) $\lambda(Q) = 2$

**Ans. (c) :** The set {x} is measurable and for any $\varepsilon > 0, \{x\} \subset \left(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}\right)$.

Therefore

$$0 \le \lambda(\{x\}) \le \lambda\left(\left(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}\right)\right) = x+\frac{\varepsilon}{2} - \left(x-\frac{\varepsilon}{2}\right) = \varepsilon$$

So $0 \le \lambda(\{x\}) \le \varepsilon$.

Since $\varepsilon > 0$ was arbitrary we conclude that λ({x})=0

Now because the set of rational numbers are countable, write $Q = \bigcup_{i=1}^{\infty}\{q_i\}$, so $\lambda(Q) = \sum_{i=1}^{\infty}\lambda(\{q_i\}) = \sum_{i=1}^{\infty} 0 = 0$

Thus, Lebesgue measure of the rational numbers Q is zero.

**62. If S denotes the set of irrational numbers in [0,1] and λ is Lebesgue measure on R, then λ(S) is equal to:**

**यदि S,[0,1] में स्थित सभी अपरिमेय संख्याओं का समुच्चय है और λ,R पर लबेग मापक है, तो λ(S) बराबर है:**

(a) 1 (b) 0
(c) ∞ (d) $\frac{1}{2}$

**Ans. (a) :** Since the set [0,1] is the disjoint union of rationales and irrationals in [0,1] and Lebesgue measure of Q ≡ set of rational numbers is 0.

λ ([0, 1]\Q) = 1 – 0 = 1 (because λ ([0, 1]) = 1)

Thus, Lebesgue measure of set of irrational S in [0,1) is 1.

**63. Let f:[0,1] → R, be a function defined by**
**f(x)= 1if x is a rational number in [0,1]**
**= 0 if x irrational number in [0,1]**
**then, the function f is:**

**मान लिया कि f:[0,1] → R, इस प्रकार परिभाषित है कि f(x)= 1, यदि x,[0,1] में स्थित परिमेय संख्या है**
**= 0 यदि x,[0,1] में स्थिति अपरिमेय संख्या है**
**तो फलन f है:**

(a) Continuous/सतत्
(b) Riemann Integrable/रीमान समाकलनीय
(c) Lebesgue Integrable/लबेग समाकलनीय
(d) Non-measurable/अमापनीय

**Ans. (c):** Clearly f(x) is not continuous because both Q and I (the set of irrationals) are dense in the real line, it follows that for any $z \in R$ we can find sequences $(x_n) \subseteq Q$ and $(y_n) \subseteq I$ such that

$$\lim x_n = \lim y_n = z \ \& \ \lim g(x_n) \neq \lim g(y_n).$$

Now if P is some partition of [0, 1], then the density of the rationales in R implies that every subinterval of P will contain a point where f(x) = 1. It follows that U (f, P) = 1. On the other hand, L (f, P) = 0 because the irrationals are also dense in R. Because this it the case for every partition P, we see that the upper integral U (f) = 1 and the lower integral L (f) = 0. The two are not equal, so we conclude that function is not Riemann integrable.

We have Lebesgue measure

$\lambda = (Q \cap [0, 1]) = 0$ and $\lambda([0, 1]\backslash Q) = 1$

Now, Lebesgue integral is given by

$$\int_{[0,1)} f d\lambda = 1.\lambda(\{Q \cap [0,1]\} + 0.\lambda([0,1] \setminus Q)$$

= 1.0+0.1

= 0

(Because $\lambda(\{Q \cap [0,1]\}) = 0$

$\lambda(\{[0,1]/Q\}) = 1)$

**64. If h is the interval of differencing then which of the following is true for the shift sperator E?**
**यदि h भेदीकरण (differencing) का अन्तराल हो, तो स्थान-परिवर्तन (shift) संकारक E के लिए निम्नलिखित में कौन सा सत्य है?**

(a) $E^n f(x)=f(x+n)$ (b) $E^n f(x)=f(x^n)$
(c) $E^n f(x)=f(x+nh)$ (d) $E^n f(x)=f(x-nh)$

**Ans. (c) :** The shift operator for a function f(x) is defined by

$$Ef(x) = f(x + h)$$

where h is the interval of differencing.

Then we have $E^n f(x) = f(x + nh)$

**65. The velocity components (u,v) in a two-dimensional fluid flow, in terms of stream function ψ , is given by:**
**प्रवाह दिशा फलन ψ के पदों में द्विआयामी तरल प्रवाह में वेग घटक (u,v) हैं:**

(a) $u = -\frac{\partial\psi}{\partial x}, v = -\frac{\partial\psi}{\partial y}$

(b) $u = \frac{\partial\psi}{\partial x}, v = -\frac{\partial\psi}{\partial y}$

(c) $u = -\frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x}$

(d) $u = -\frac{\partial\psi}{\partial y}, v = \frac{\partial\psi}{\partial x}$

**Ans. (d) :** For a two dimensional incompressible flow parallel to the xy-plane in a rectangular Cartesian coordinate system, let u and v denote the velocity vectors in the x and y direction respectively.

Equation of continuity : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial v}{\partial y} = \frac{\partial(-u)}{\partial x}$

Equation of lines of flow or streamline is

$\frac{dx}{v} = \frac{dy}{v}$ or $vdx - udy = 0$

Which shows that equation of streamline must be an exact differential, $d\psi$(say). Then, we have

$$vdx - udy = \left(\frac{\partial\psi}{\partial x}\right)dx + \left(\frac{\partial\psi}{\partial y}\right)dy$$

so $u = -\frac{\partial\psi}{\partial y}$ and $v = \frac{\partial\psi}{\partial x}$

ψ us stream function

**66. Which of the following statements is true in case of linear programming?**
**निम्नलिखित में से कौन-सा कथन रैखिक प्रोग्रामन के मामले मे ंसत्य है?**

(a) An optimal solution exists at extreme points of a set of feasible solutions./सम्भाव्य हल के समूह के चरम बिन्दु पर एक अनुकूलतम हल मौजूद होता है।
(b) An optimal solution gives a hyperplane which is a supporting hyperplane to the set of feasible solutions./अनुकूलतम हल का हाइपरप्लेन सम्भाव्य हल के समुच्चय का सपोर्टिंग हाइपरप्लेन होता है।
(c) Extreme points and basic feasible solutions are in one-one correspondence./चरम बिन्दु व मूलभूत सम्भाव्य हल एकैकी होते हैं।
(d) A set of feasible solutions is not necessarily a convex set./सम्भाव्य हल के समुच्चय का उत्तर समुच्चय होना आवश्यक नहीं हैं।

**Ans. (a) :** Let R be the feasible region for a Linear programming problem then an optimal solution exists at extreme points/corner points of the set of feasible solutions

**67. Consider the following linear programming problem:**
**Maximize:**

$$\mathbf{z = 3x_1 - 2x_2}$$

**Subject to:**

$$\mathbf{x_1 + x_2 \leq 1}$$
$$\mathbf{2x_1 + 2x_2 \geq 4}$$
$$\mathbf{x_1, x_2 \geq 0}$$

**Which one of the following statements is true?**
**निम्नलिखित रैखिक प्रक्रमन समस्या पर विचार कीजिएः**
**अधिकतमः**

$$\mathbf{z = 3x_1 - 2x_2}$$

**के अधीनः**

$$\mathbf{x_1 + x_2 \leq 1}$$
$$\mathbf{2x_1 + 2x_2 \geq 4}$$
$$\mathbf{x_1, x_2 \geq 0}$$

**निम्नलिखित में से कौन सा कथन सत्य है?**

(a) (0,0) is an optimal solution/(0,0) एक अनुकूलतम समाधान है।
(b) (1,0) is an optimal solution/(1,0) एक अनुकूलतम समाधान है।
(c) Solutions are unbounded/समाधान असीमित है।
(d) The constraints are inconsistent/प्रतिबन्ध (constraints) असंगत है।

**Ans. (d) :** Because given constraints are inconsistent as given by

$x_1 + x \le 1 \text{ and } x_1 + x_2 \ge 2$

maximize

$Z = 3x_1 - 2x_2$

Subject to,

$x_1 + x_2 \le 1$ .............(1)

$2x_1 + 2x_2 \ge 4$ ..........(II)

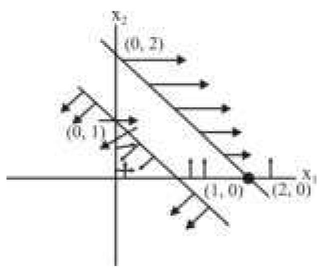
$x_1, x_2 \ge 0$

equation (I)

| | | |
|---|---|---|
| $x_1$ | 0 | 1 |
| $x_2$ | 1 | 0 |

equation (II)

| | | |
|---|---|---|
| $x_1$ | 0 | 2 |
| $x_2$ | 2 | 0 |




The problem is not well-defined.

**68. The dual of the following linear programming problem:**
**Maximize:**

$$Z = C^t X$$
$$AX = b$$
$$X \ge 0$$

**is :**

**निम्नलिखित रैखिक प्रक्रमन समस्याः**
**अधिकतमः**

$$Z = C^t X$$
$$AX = b$$
$$X \ge 0$$

**का द्वैत है:**

(a) Max $Z = b^t X, AX=C, X\ge 0$/ अधिकतम $Z = b^t X, AX=C, X\ge 0$

(b) Max $Z= b^tY, A^tX\ge C^t$, Y free/अधिकतम $Z= b^t Y, A^tX\ge C^t$, Y free

(c) Min $Z= b^tY, A^tY\ge C^t$, Y free /अधिकतम= $b^t Y, A^tY\ge C^t$, Yfree

(d) Min $Z= b^tY, A^tY\ge C^t, Y\ge 0$ free /अधिकतम $Z= b^t Y, A^tY\ge C^t, Y\ge 0$ free

**Ans. (c) :** The dual of the given linear programming problem takes the form

Minimize $\quad Z = b^tY$

Subject to $\quad A^tY\ge C^t$

Y free

Where $b^t, A^t, C^t$ are the transposes of b,A,C.

**69. If $\omega \ne 1$ is a cube root of unity, then:**

$$1+\omega+\omega^2+.............+\omega^{3n-1} =$$

**यदि $\omega \ne 1$ इकाई का घनमूल है, तो:**

$$1+\omega+\omega^2+.............+\omega^{3n-1} =$$

(a) 0 (b) 1
(c) 3n (d) n–1

**Ans. (a) :** If $\omega \ne 1$ is a cube root of unity, then

$$1 + \omega + \omega^2 = 0$$

Now

$1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+......+\omega^{3n-1}$

$\Rightarrow 0+\omega^3(1+\omega+\omega^2)+......+\omega^{3n-3}(1+\omega+\omega^2)$

$= 0$

**70. Which of the following functions are analytic on C?**

**(i) $f(z) = \bar{z}$**
**(ii) g(z) =a polynomial in z**
**(iii) h(z) = tan z**
**(iv) u(z) = cosh z**

**निम्नलिखित में से कौन सा फलन पर विश्लेषिक है?**

**(i) $f(z) = \bar{z}$**
**(ii) g(z) = a polynomial in z**
**(iii) h(z) = tan z**
**(iv) u(z) = cosh z**

(a) (i) (b) (iii)
(c) (iv)
(d) (ii) and (iv)/ (ii) और (iv)

**Ans. (d) :** Let f(z) = u (x, y) + iv (x, y); $\forall$ z = x + iy $\in D\subseteq C$ is on open set. Assume that u & v have continuous first order partial derivatives throughout D and that they satisfy Cauchy- Riemann equations at z $\in$ D. Then f (z) exists.

- Polynomial functions in z $\in$C are analytic in a given region
- u(z) = cos hz is analytic in given region.
- $f(z)=\bar{z}$ is non-analytic in a given region because Cauchy- Riemann equations are not satisfied as

$f(z)=\bar{z} \Rightarrow f(x+iy)=x-iy$

$\Rightarrow u = Re(f) = x; v = Im(f) = -y$

$\therefore u_x = 1, v_x = 0, u_y = 0, v_y = -1 \Rightarrow u_x \ne v_y$

at any point (x,y) in the region

- $h(z) = \tan z := \dfrac{\sin z}{\cos z} = -i\dfrac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$

The differentiation rules for analytic functions then allow to conclude that tan is analytic in all points z $\in$ C where $e^{iz} + e^{-iz} \ne 0$.

Now $e^{iz} + e^{-iz} = -e^{-iz}(e^{i(2z-\pi)}-1)$ is equal to 0 iff

$$i(2z - \pi) = 2k\pi i, k\in Z,$$

$$z = k\pi + \frac{\pi}{2}, k \in Z.$$

# Uttar Pradesh Public Service Commission
# Assistant Professor 2021
# Solved Paper

**[Exam Date-15.03.2022]**

**1. If the Nijenhuis tensor vanishes on an almost complex manifold, then the F-connection is**
**यदि एक लगभग सम्मिश्र बहुमुख पर निजेनहुअस प्रदिश शून्य हो जाता है, तो F-संयोजन**

(a) Symmetric/सममित है
(b) Half-symmetric/अर्ध-सममित है
(c) Anti-symmetric/प्रति-सममित है
(d) Not necessarily symmetric/का सममित होना आवश्यक नहीं है

**Ans. (d)** If the Nijenhuis tensor vanishes on an almost complex manifold, it means that the manifold can be endowed with a complex structure. This implies that the manifold is integrable, and a complex structure can be applied to it.
However, even in this case, the F-connection, which is the connection preserving the almost complex structure, is not necessarily symmetric Thus, when the Nijenfuis tensor vanishes, it is not required for the F-connection to be symmetric.

**2. For term by term integration of an infinite series of integrable functions, the condition of uniform convergence of the series is**
**किसी समाकलनीय फलनों की अनन्त श्रेणी को पदशः समाकलन करने के लिये श्रेणी का एकसमान अभिसरित होने का प्रतिबंध है**

(a) Necessary and sufficient/आवश्यक एवं पर्याप्त
(b) Necessary but not sufficient/आवश्यक परन्तु पर्याप्त नहीं
(c) Sufficient but not necessary/पर्याप्त परन्तु आवश्यक नहीं
(d) Neither sufficient nor necessary/न तो पर्याप्त न ही आवश्यक

**Ans. (c) :** If a series $\sum_{n=1}^{\infty} f_n(x)$ converges to f(x) uniformly on the interval [a, b] and also if each $f_n(x)$ is integrable then f(x) is integrable on [a, b] and $\sum_{n=1}^{\infty}\int_a^b f_n(x)dx = \int_a^b \sum_{n=1}^{\infty} f_n(x)dx = \int_a^b f(x)\,dx$ and for term by term integration of an infinite series of integrable functions, the condition of uniform convergence of the series is sufficient but not necessary.

**3. Which of the following statements is/are true?**
**I: Union of two topologies on X is always a topology on X.**
**II: Intersection of two topologies on X is always a topology on X.**
**निम्नलिखित कथनों मे से कौन-सा/से कथन सत्य है/हैं?**
**I : X पर दो संस्थितियों का संघ सदैव X की एक संस्थिति होती है।**
**II: X पर दो संस्थितियों का सर्वनिष्ठ सदैव X की एक संस्थिति होती है।**

(a) Only I/केवल I
(b) Only II/केवल II
(c) Both I and II/ I एवं II दोनों
(d) Neither I nor II/न तो I न ही II

**Ans. (b) :**Let X ={a, b, c}and $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\varphi, \{b\}, X\}$ then $\tau_1$ and $\tau_2$ are topologies on X but $\tau_1 \cup \tau_2 = \{\phi, \{a\}, \{b\}, X\}$ is not a topology on X as $\{a\}, \{b\} \in \tau_1 \cup \tau_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \tau_1 \cup \tau_2$.
Therefore, union of two topologies on X need not be a topology on X.
Now Let $\tau_1$ and $\tau_2$ be two topologies on X.
To show that $\tau_1 \cap \tau_2$ is also a topology on X.
(i) $\because \phi$ and $X \in \tau_1$ as well as $\tau_2$
$\Rightarrow \phi, X \in \tau_1 \cap \tau_2$
(ii) Let $\{U_i\}_{i\in I}$ be a family of open sets such that $U_i \in \tau_1 \cap \tau_2 \ \forall\, i \in I$.
$\Rightarrow U_i \in \tau_1$ and $U_i \in \tau_2$, $\forall\, i \in I$
$\Rightarrow \cup_{i\in I} U_i \in \tau_1$ and $\cup_{i\in I} U_i \in \tau_2$
$(\because \tau_1$ and $\tau_2$ are topologies on X$)$
$\Rightarrow \cup_{i\in I} U_i \in \tau_1 \cap \tau_2$
(iii) Let $\cup_1, \cup_2, ....... \cup_n \in \tau_1 \cap \tau_2$
$\Rightarrow U_i \in \tau_1$ and $U_i \in \tau_2$, i = 1, 2, ........,n
$\Rightarrow \bigcap_{i=1}^{n} U_i \in \tau_1$ and $\bigcap_{i=1}^{n} U_i \in \tau_2$
$(\because \tau_1$ and $\tau_2$ are topologies on X$)$
$\Rightarrow \bigcap_{i=1}^{n} U_i \in \tau_1 \cap \tau_2$
Hence $\tau_1 \cap \tau_2$ is also a topology on X.

4. **Fourier sine transform of the function f (x)= $\frac{1}{x}$ is**

**फलन f (x)= $\frac{1}{x}$ का फोरियर ज्या रूपातंरण है**

(a) $\sqrt{\pi}$ (b) $\sqrt{\frac{\pi}{2}}$

(c) $\sqrt{\frac{2}{\pi}}$ (d) $\frac{\pi}{2}$

**Ans. (b) :** By Fourier sine transform.

$$F\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$\Rightarrow F\left\{\frac{1}{x}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx$$

$\because$ We have $\sqrt{\frac{2}{\pi}}\int_0^\infty\left(\frac{e^{-ax}}{x}\right)\sin sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow F\left\{\frac{1}{x}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty\frac{1}{x}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{0}\right)$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}(\infty)$$

$$= \sqrt{\frac{2}{\pi}}\times\frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$

5. **If p is a prime number, then any group of order 2p has a normal subgroup of order**

**यदि p एक अभाज्य संख्या है, तो क्रम 2p के किसी समूह के लिए क्रम का एक प्रसामान्य उपसमूह होगा**

(a) P – 1 (b) p

(c) p +1 (d) $\frac{1}{2}(p+1)$

**Ans. (b) :** Given order of group G is 2p, p is a prime number, and it has a subgroup of order p whose index is 2. And any subgroup of index 2 is normal subgroup of G.

$\Rightarrow$ It has a normal subgroup of order p.

6. **The rate of convergence of Newton-Raphson method is**

**न्यूटन-रैफ्सन विधि की अभिसारिता की दर है**

(a) Cubic/घन

(b) Fourth order/चतुर्थ क्रम

(c) Linear/रैखिक

(d) Quadratic/वर्ग

**Ans. (d) :** By Newton- Raphson method we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let α be root of $f(x)$

$\Rightarrow x_n = \alpha + \varepsilon_n$ & $x_{n+1} = \alpha + \varepsilon_{n+1}$

where $\varepsilon_n$ & $\varepsilon_{n+1}$ are errors.

$$\Rightarrow \alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

$$\Rightarrow \varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha)+\varepsilon_n f'(\alpha)+\frac{\varepsilon_n^2}{2!}f''(\alpha)+......}{f'(\alpha)+\varepsilon_n f''(\alpha)+\frac{\varepsilon_n^2}{2!}f'''(\alpha)+......}$$

Since higher power of $\varepsilon_n$ are very small, so on neglecting them we have

$$\varepsilon_{n+1} \approx \varepsilon_n - \frac{f(\alpha)+\varepsilon_n f'(\alpha)+\frac{\varepsilon_n^2}{2!}f''(\alpha)}{f'(\alpha)+\varepsilon_n f''(\alpha)}$$

$$\varepsilon_{n+1} \approx \frac{\varepsilon_n^2 f''(\alpha)-0-\frac{1}{2}\varepsilon_n^2 f''(\alpha)}{f'(\alpha)+\varepsilon_n f''(\alpha)} \qquad (\because f(\alpha)=0)$$

$$\varepsilon_{n+1} \approx \frac{\varepsilon_{n^2}}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$\Rightarrow$ Rate of Convergence is quadratic.

7. **If $\phi_1$ and $\phi_2$ are arbitrary functions, then the solution of the partial differential equation $\gamma = a^2 t$, is**

**यदि $\phi_1$ एवं $\phi_2$ स्वेच्छ फलन हो, तो आंशिक अवकल समीकरण $\gamma = a^2 t$ का हल है**

(a) $z = \phi_1(y+ax)+\phi_2(y-ax)$

(b) $z = \phi_1(y)+\phi_2(y-ax)$

(c) $z = \phi_1(y+ax)+\phi_2(y)$

(d) $z = \phi_1(x)+\phi_2(y)+axy$

**Ans. (a) :** Given $\gamma = a^2 t$ ...(i)

We have $\gamma = \frac{\partial^2 z}{\partial x^2}$ and $t = \frac{\partial^2 z}{\partial y^2}$

Putting in (i), we get

$$\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$$

$\Rightarrow (D^2 - a^2(D')^2)z = 0$ where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

For Auxiliary equation put D = m and $D' = 1$

$m^2 - a^2 = 0 \Rightarrow m = \pm a$

$\Rightarrow$ Its general solution is

$$\boxed{z = \phi_1(y+ax)+\phi_2(y-ax)}$$

where $\phi_1$ and $\phi_2$ are arbitrary functions.

**8. Let f: [a, b]→ R be a bounded function and P be a partition of [a, b], then U (P, – f) =**
**मान लीजिए कि f : [a, b]→ R एक परिबद्ध फलन है तथा P, [a, b] का एक विभाजन है, तब U (P, – f) =**

(a) U (P, f) (b) –U (P, f)
(c) – L(P, f) (d) L (P, f)

**Ans. (c) :** Let $P = \{a = x_0 \le x_1 \le x_2 \le ....... \le x_n = b\}$ be the partition of [a, b] then upper Riemann sum of f is defined as

$$U(P, f) = \sum_{i=1}^{n} M_i \ \Delta x_i$$

where $M_i$ is supremum of f in $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$; i = 1, 2, ...,n
and lower Riemann sum of f is defined as

$$L(P, f) = \sum_{i=1}^{n} m_i \ \Delta x_i$$

where $m_i$ is infimum of f in $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$ ; i = 1, 2,....,n

Consider $U(P, -f) = \sum_{i=1}^{n} M'_i \ \Delta x_i$

where $M'_i$ is supermum of –f in $[x_{i-1}, x_i]$ which implies that $-M'_i$ is infimum of f in $[x_{i-1}, x_i]$.

Hence, $U(P, -f) = \sum_{i=1}^{n} M'_i \ \Delta x_i = -\sum_{i=1}^{n} -M'_i \ \Delta x_i = -L(P, f)$.

**9. The dimension of Lie group SL (n, C) is**
**ली-समूह SL (n, C) की विमा है**

(a) $n^2$ (b) $n^2 - 1$
(c) $2(n^2 - 1)$ (d) $2n^2$

**Ans. (c) :** SL (n, c) = $\{A \in GL(n, c) \mid |A| = 1\}$
Since |A| = 1 condition gives one constraint on a complex number so there are $2(n^2 - 1)$ arbitrary entry
$\therefore$ dim (SL(n, c)) = $2(n^2 - 1)$

**10. Which of the following set of vectors is a basis for vector space $R^3$?**
**निम्नलिखित मे कौन-सा सदिशों का समुच्चय सदिश समष्टि $R^3$ का एक आधार है?**

(a) $\{(1,-1,1), (1,0,2), (2,-1,3)\}$
(b) $\{(1,-1,1),(1,0,2),(2,1,1)\}$
(c) $\{(1,-1,1),(1,0,2),(0,1,1)\}$
(d) $\{(1,-1,1),(1,0,2),(1,-2,0)\}$

**Ans. (b) :** (a) Vectors are not linearly independent as (1, –1, 1) + (1, 0, 2) = (2, –1, 3).

(b) Consider $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}$; $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{pmatrix}; R_3 \to R_3 - 3R_2$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

therefore, the set {(1, –1, 1), (1, 0, 2), (2, 1, 1)} is linearly independent set and generates $R^3$ because this set has 3 elements which is same as dimension of $R^3$. Hence, it is a basis.
(c) vectors are also not linearly independent as (1, 0, 2) – (1, –1, 1) = (0, 1, 1).
(d) vectors are also not linearly independent as (1, 0, 2) + (1, –2, 0) = (2, –2, 2) = 2 (1, –1, 1).

**11. The value of $L^{-1}\left\{\frac{e^{-1/s}}{s}\right\}$ is**

**$L^{-1}\left\{\frac{e^{-1/s}}{s}\right\}$ का मान है**

(a) $J_0(2t)$ (b) $J_0\left(2\sqrt{t}\right)$
(c) $J_0\left(\sqrt{2}\ t\right)$ (d) $J_0\left(\sqrt{2t}\right)$

**Ans. (b) :** The Bessel function of first kind of order zero given as $J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n} (n!)^2}$

and $L^{-1}\left\{\frac{e^{-1/s}}{s}\right\} = J_0\left(2\sqrt{t}\right)$

**12. Which of the following is a topology on X = {a, b, c, d}?/निम्नलिखित में से कौन X = {a, b, c, d} पर एक संस्थिति है?**

(a) $\{\phi, \{a\}, \{b\}, \{b, c\}, X\}$
(b) $\{\phi, \{a\}, X\}$
(c) $\{\phi, \{a\}, \{b\}, X\}$
(d) $\{\phi, \{c\}, \{d\}, \{b, c\}, X\}$

**Ans. (b) : Definition:** Let X be a set and $\tau$ be the collection of subsets of X. The $\tau$ is called a topology on X if-
(1) $\phi$ And X are in $\tau$.
(2) The arbitrary union of elements of $\tau$ are also in $\tau$.
(3) The finite intersection of elements of $\tau$ are also in $\tau$.
In option (a) $\{a\} \cup \{b\} = \{a, b\} \notin \{\phi, \{a\}, \{b\}, \{b, c\}, X\}$
$\Rightarrow \{\phi, \{a\}, \{b\}, \{b, c\}, X\}$ is not a topology on X.
In option (b) consider $\tau = \{\phi, \{a\}, X\}$
(1) $\phi$ And X are in $\tau$.
(2) Arbitrary union of elements of $\tau$ are also in $\tau$.
(3) The finite intersection of elements of $\tau$ are also in $\tau$.
In option (c) $\{a\} \cup \{b\} = \{a, b\} \notin \{\phi, \{a\}, \{b\}, X\}$
$\Rightarrow \{\phi, \{a\}, \{b\}, X\}$ is not a topology on X.
In option (d) $\{c\} \cup \{d\} = \{c, d\} \notin \{\phi, \{c\}, \{d\}, \{b, c\}, X\}$
$\Rightarrow \{\phi, \{c\}, \{d\}, \{b, c\}, X\}$ is not a topology on X.

13. **The order of the difference equation $\Delta^2 y_x + 3\Delta y_x - 3y_x = x$, is**
**अंतर समीकरण $\Delta^2 y_x + 3\Delta y_x - 3y_x = x$, की कोटि हैं**
(a) 1 (b) 2
(c) 3 (d) 4

**Ans. (b) :** Since $\Delta y_x = y_{x+1} - y_x$ .... (1)
$\Rightarrow \Delta^2 y_x = \Delta(\Delta y_x) = \Delta(y_{x+1} - y_x)$
$= (y_{x+2} - y_{x+1}) - (y_{x+1} - y_x)$
$\Rightarrow \Delta^2 y_x = y_{x+2} - 2y_{x+1} + y_x$ .... (2)
Using (1) and (2) given difference equation becomes
$\Delta^2 y_x + 3\Delta y_x - 3y_x = x$
$\Rightarrow y_{x+2} - 2y_{x+1} + y_x + 3y_{x+1} - 3y_x - 3y_x = x$
$\Rightarrow y_{x+2} + y_{x+1} - 5y_x = x$
and order of the difference equation is the difference between the largest and smallest argument appearing in difference equation divided by unit of increment
$\Rightarrow$ Order $= \frac{x+2-x}{1} = 2$

14. **General solution x = z (x, y) of the partial differential equation $y^2zp + x^2zq = xy^2$, is**
**आंशिक अवकल समीकरण $y^2zp + x^2zq = xy^2$ का व्यापक हल x = z (x, y) है**
(a) $F(x^3 - y^3, x^2 - z^2) = 0$
(b) $F(x^3 - y^3, x^2 - z^2) = 0$
(c) $F(x^3 + y^3, x^2 + z^2) = 0$
(d) $F(x^3 + y^3, x^2 - z^2) = 0$

**Ans. (a) :** Lagrange's Auxiliary equation is
$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$
Consider $\frac{dx}{y^2z} = \frac{dy}{x^2z}$
$\Rightarrow x^2\,dx = y^2\,dy$
$\Rightarrow x^3 - y^3 = C_1$ : $C_1$= arbitrary constant
also $\frac{dx}{y^2z} = \frac{dz}{xy^2}$
$\Rightarrow xdx = zdz$
$\Rightarrow x^2 - z^2 = C_2$ : $C_2$ = arbitrary constant
$\therefore$ General solution will be
$F(x^3 - y^3, x^2 - z^2) = 0$

15. **Let N be a normed linear space and x, y $\in$ N, then/मान लीजिये N एक मानकित रैखिक समष्टि है और x, y $\in$ N, तो**
(a) $\big|\|x\| - \|y\|\big| \le \|x - y\|$
(b) $\big|\|x\| - \|y\|\big| < \|x - y\|$
(c) $\big|\|x\| - \|y\|\big| \ge \|x - y\|$
(d) $\big|\|x\| - \|y\|\big| > \|x - y\|$

**Ans. (a) :**
$\|x\| = \|(x-y) + y\| \le \|x-y\| + \|y\|$
$\Rightarrow \|x\| - \|y\| \le \|x-y\|$
and $\|y\| = \|(y-x) + x\| \le \|y-x\| + \|x\|$
$\Rightarrow \|y\| - \|x\| \le \|x-y\|$
$\Rightarrow \big|\|x\| - \|y\|\big| \le \|x-y\|$

16. **Inner product of tensors $A^{ij}$ and $B_{hk}$ will be a mixed tensor of the type/प्रदिशों $A^{ij}$ और $B_{hk}$ का आंतर गुणन एक मिश्रित प्रदिश होगा**
(a) (2, 1)/ (2, 1) प्रकार का
(b) (1, 2)/ (1, 2) प्रकार का
(c) (2, 2)/ (2, 2) प्रकार का
(d) (1, 1)/ (1, 1) प्रकार का

**Ans. (d) :** Inner product of two tensors is obtained by first taking outer product and then contracting the outer product. Now outer product of contravariant tensor $A^{ij}$ of rank 2 and covariant tensor $B_{hk}$ of rank 2 gives a mixed tensor of type (2, 2) which on contraction gives a mixed tensor of the type (1,1).

17. **Let X = {a, b, c} and T = {ϕ, {a}, {a, b},{a, c}, X} be a topology on X. If A = {a, c}, then derived set of A is**
**माना X = {a, b, c} तथा T = {ϕ, {a}, {a, b},{a, c}, X} X पर एक संस्थिति है। यदि A = {a, c}, तो A का व्युत्पन्न समुच्च है**
(a) ϕ (b) {b, c}
(c) {a, c} (d) {a, b}

**Ans. (b) :** Given X = {a, b, c,} and T = {ϕ, {a}, {a, b}, {a, c}, X} is a topology on X.
If A = {a, c} then derived set A' of A is the set of all limit points of A.
Now 'a' is not a limit point of A since {a} is a neighbourhood of 'a' which does not contain any other point of A other than 'a' itself. Whereas 'b' and 'c' are limit points of A since {a, b} and X are neighbourhoods of 'b' containing a point of A other than 'b' and {a, c} and X are neighbourhoods of 'c' containing a point of A other than 'c'
$\therefore$ A' = {b, c}

18. **For all a, b in Boolean algebra, the value of (a + b) a' b' is**
**बूलियन बीजगणित में सभी a, b के लिये, (a + b) a' b' का मान है**
(a) a (b) 1
(c) a + b (d) 0

**Ans. (d) :**

| a | b | a' | b' | a + b | a'b' | (a + b)a'b' |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 |

$\Rightarrow$ (a + b) a'b' = 0

**19. Newton-Raphson iterative formula to obtain the cube root of N is /N का घनमूल ज्ञात करने के लिये न्यूटन-रैफ्सन का पुनरावृत्तीय सूत्र है**

(a) $x_{i+1}=\frac{1}{3}\left(2x_i-\frac{N}{x_i^2}\right)$ (b) $x_{i+1}=\frac{1}{3}\left(2x_i+\frac{N}{x_i^2}\right)$

(c) $x_{i+1}=\frac{1}{3}\left(2x_i+\frac{N}{x_i}\right)$ (d) $x_{i+1}=\frac{1}{3}\left(2x_i-\frac{N}{x_i}\right)$

**Ans. (b) :** Let x be cube root of N

$\Rightarrow\ x=(N)^{1/3}$

$\Rightarrow\ x^3-N=0$

Consider $f(x)=x^3-N$

or $f(x_i)=x_i^3-N$

$\Rightarrow\ f'(x_i)=3x_i^2$

Newton – Raphson iterative formula is given as

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

$$\Rightarrow x_{i+1}=x_i-\frac{x_i^3-N}{3x_i^2}$$

$$\Rightarrow x_{i+1}=\frac{3x_i^3-x_i^3+N}{3x_i^2}$$

$$\Rightarrow x_{i+1}=\frac{2x_i^3+N}{3x_i^2}$$

$$\Rightarrow x_{i+1}=\frac{1}{3}\left(2x_i+\frac{N}{x_i^2}\right)$$

**20. The set of points in the interval [2, 4] and in interval (1, 2) are**
**अंतराल एवं अंतराल के बिन्दुओं के समुच्चय होते हैं**

(a) Finite /परिमित
(b) Cardinally equivalent/कार्डिनल रूप से समकक्ष
(c) Cardinally unequivalent /कार्डिनल रूप से असमकक्ष
(d) None of the above/उपरोक्त मे से कोई नहीं

**Ans. (b) :** Both the interval [2, 4] and (1, 2) are uncountable having cardinality 'Continuum' therefore they are cardinally equivalent.

**21. Given K.E. = T = $\frac{1}{2}m\dot{r}^2$ and P.E. = V= $\frac{1}{2}\left(1+\frac{\dot{r}^2}{c^2}\right)$, then Hamiltonian H = ?**

**दिया है K.E. = T = $\frac{1}{2}m\dot{r}^2$ तथा P.E. = V = $\frac{1}{2}\left(1+\frac{\dot{r}^2}{c^2}\right)$, तो हैमिल्टोनियन H = ?**

(a) T + V (b) $T+V+\frac{2\dot{r}^2}{c^2}$

(c) $T+V-\frac{2\dot{r}^2}{rc^2}$ (d) $T+V+\frac{2\dot{r}^2}{rc^2}$

**Ans. (*) :** Given $T=\frac{1}{2}m\,\dot{r}^2$ and $V=\frac{1}{2}\left(1+\frac{\dot{r}^2}{c^2}\right)$

Now Lagrangian L = T – V

$$\Rightarrow\quad L=\frac{1}{2}m\dot{r}^2-\frac{1}{2}\left(1+\frac{\dot{r}^2}{c^2}\right)\quad .....(i)$$

and Hamiltonian H is defined as

$$H=-L+\sum_{r=1}^{n}p_r\dot{q}_r\quad .....(ii)$$

Where $p_r$ = generalized momentum of particle
and $q_r$ = generalized co-ordinates of particle.

and $p_r=\frac{\partial L}{\partial\dot{q}_r}$ Here $q_r = r$

$$\Rightarrow p_r=\frac{\partial L}{\partial\dot{r}}=\frac{\partial}{\partial\dot{r}}\left(\frac{1}{2}m\dot{r}^2-\frac{1}{2}\left(1+\frac{\dot{r}^2}{c^2}\right)\right)\quad (\text{From (i)})$$

$$\Rightarrow p_r=m\dot{r}-\frac{\dot{r}}{c^2}$$

Putting these values in (ii), we get

$$H=-\left(\frac{1}{2}m\dot{r}^2-\frac{1}{2}\left(1+\frac{\dot{r}^2}{c^2}\right)\right)+\left(m\dot{r}-\frac{\dot{r}}{c^2}\right)\dot{r}$$

$$H=\frac{1}{2}m\dot{r}^2+\frac{1}{2}\left(1+\frac{\dot{r}^2}{c^2}\right)-\frac{\dot{r}^2}{c^2}$$

$$\boxed{H=T+V-\frac{\dot{r}^2}{c^2}}$$

**22. Let Q be the field of rational numbers. Then over Q, $\sqrt{2}+\sqrt{3}$ is algebraic of degree**
**मान लीजिए Q परिमेय संख्याओं का क्षेत्र है। तब $\sqrt{2}+\sqrt{3}$ , Q घात का बीजगणितीय है**

(a) 2 (b) 3
(c) 4 (d) 6

**Ans. (c) :** $\because\left(\sqrt{2}+\sqrt{3}\right)^2=2+3+2\sqrt{6}=5+2\sqrt{6}$

and

$$\left(\sqrt{2}+\sqrt{3}\right)^4=\left(5+2\sqrt{6}\right)^2=25+24+20\sqrt{6}=49+20\sqrt{6}$$

$$\Rightarrow\left(\sqrt{2}+\sqrt{3}\right)^4=50+20\sqrt{6}-1=10\left(5+2\sqrt{6}\right)-1$$

$\Rightarrow \left(\sqrt{2}+\sqrt{3}\right)^4 - 10\left(5+2\sqrt{6}\right)+1=0$

$\Rightarrow \left(\sqrt{2}+\sqrt{3}\right)^4 - 10\left(\sqrt{2}+\sqrt{3}\right)^2 + 1=0$

$\Rightarrow \sqrt{2}+\sqrt{3}$ Satisfies a polynomial

(let $\sqrt{2}+\sqrt{3} = x$ )

$x^4 - 10x^2 + 1 = 0$ over Q which is of degree 4. It is also a minimal polynomial for $\left(\sqrt{2}+\sqrt{3}\right)$

Therefore $\left(\sqrt{2}+\sqrt{3}\right)$ is algebric of degree 4.

**23. The curve, for which $\int_0^{\pi/2}\left(y'^2+2xyy'\right)dx$ subjected to y (0) = 1, y $\left(\frac{\pi}{2}\right)$ = 1 is extremised, is**

**वक्र, जिसके लिये $\int_0^{\pi/2}\left(y'^2+2xyy'\right)dx$, y(0) = 1, y $\left(\frac{\pi}{2}\right)$ = 1 के आधीन चरमीकरण है, है**

(a) y = 2 sinx + cosx (b) y = sinx + 2 cosx
(c) y = sinx – cosx (d) y = sinx + cosx

**Ans. (d) :** Here functional F(x, y, y') = $y'^2 + 2xyy'$

$\therefore$ By Euler's equation we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$\Rightarrow 2xy' - \frac{d}{dx}\left(2y'+2xy\right) = 0$

$\Rightarrow 2xy' - 2y'' - 2(xy' + y) = 0$

$\Rightarrow y'' + y = 0$

Auxiliary equation is

$m^2 + 1 = 0$

$\Rightarrow m = \pm i$

$\Rightarrow y(x) = C_1 \cos x + C_2 \sin x$

$y(0) = 1 \Rightarrow 1 = C_1$

$y\left(\frac{\pi}{2}\right) = 1 \Rightarrow 1 = C_2$

$\therefore y(x) = \cos x + \sin x = \sin x + \cos x$

**24. For f (z) = $\frac{e^z \sinh z}{z^4}$, the residue at z = 0 is**

**f (z) = $\frac{e^z \sinh z}{z^4}$ के लिये, z = 0 पर अवशेष है**

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{1}{6}$

**Ans. (a) :** Given f(z) = $\frac{e^z \sinh z}{z^4}$

$$= \frac{e^z}{z^4}\times\frac{\left(e^z - e^{-z}\right)}{2} \quad \left(\because \sinh z = \frac{e^z - e^{-z}}{2}\right)$$

$$= \frac{1}{2}\frac{e^{2z}-1}{z^4}$$

$$= \frac{1}{2z^4}\left[1+2z+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+.....-1\right]$$

$$= \frac{1}{z^3}+\frac{1}{z^2}+\frac{2}{3}\frac{1}{z} + ........$$

Now residue at z = 0 is coefficient of $\frac{1}{z}$ in Laurent series expansion of f(z) about z = 0 is = 2/3

**25. If two bounded lattices A and B are complemented, then which of the following is complemented?/यदि दो परिबद्ध जाली A और B पूरक है, तब निम्नलिखित में से कौन पूरक है?**

(a) A – B (b) A + B
(c) A × B
(d) All of the above/उपरोक्त सभी

**Ans. (c):** A lattice L is complemented if every element $a \in L$ has a complement $b \in L$, such that:

$a \vee b = 1$ and $a \wedge b = 0$ where 1 and 0 are the greatest and least elements of the lattice, respectively.

- A and B are two bounded complemented lattices
- A = B: The difference operation between two lattices doesn't necessarily preserve the complemented property.
- A + B: The direct sum of two lattice does not always result in a complemented lattice unless certain conditions are met.
- AB: The Cartesian product (or direct product) of two complemented lattices is always complemented. This is because the complement of an element $(a, b) \in AB$ is (a', b'), where a' and b' are complements of a in A and b in B, respectively.

The Cartesian product AB of two complemented lattices is always complemented.

Hence, the answer is (c).

**26. If $P_n(x)$ denotes Legendre's polynomial, then the value of $P'_{n+1}(x) - xP'_n(x)$ is equal to**

**यदि $P_n(x)$ लेजेन्ड्रीज बहुपद को दर्शाता है, तो $P'_{n+1}(x) - xP'_n(x)$ का मान बराबर है**

(a) $(n + 1) P_n(x)$ (b) $n^2$
(c) $P_n(x)$ (d) $P'_n(x)$

**Ans. (a) :** By recurrence relations we have

$n P_n(x) = x P'_n(x) - P'_{n-1}(x)$ ....(i)

and $(2n + 1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$ ....(ii)

Subtract (ii) from (i), and we get

$(n + 1) P_n(x) = P'_{n+1}(x) - xP'_n(x)$

$\Rightarrow P'_{n+1}(x) - xP'_n(x) = (n + 1) P_n(x)$

**27. Unique polynomial f(x) of degree 2 or less such that f (0) = 1, f(1) = 3 and f(3) = 55 is**

**2 या 2 से कम डिग्री का एकमात्र बहुपद f(x) जिसके लिये f (0) = 1, f(1) = 3 और f(3) = 55 हो, है**

(a) $1 + 6x - 8x^2$ (b) $1 - 6x + 8x^2$

(c) $1 + 6x + 8x^2$ (d) $1 - 6x - 8x^2$

**Ans. (b) :**

Given

| x | 0 | 1 | 3 |
|---|---|---|---|
| f(x) | 1 | 3 | 55 |

By Lagrange's Interpolation formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$\Rightarrow f(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} \times 1 + \frac{(x-0)(x-3)}{(1-0)(1-3)} \times 3 + \frac{(x-0)(x-1)}{(3-0)(3-1)} \times 55$$

$$\Rightarrow f(x) = \frac{x^2-4x+3}{3} + \frac{x^2-3x}{(-2)} \times 3 + \frac{x^2-x}{6} \times 55$$

$$\Rightarrow f(x) = \frac{x^2-4x+3}{3} + \frac{3x^2-9x}{-2} + \frac{55x^2-55x}{6}$$

$\Rightarrow f(x) = 1 - 6x + 8x^2$

**28. If T : $R^2$ (R) → $R^2$ (R) is defined by T(2, 3) = (4, 5) and T (1, 0) = (0, 0), then T (x, y) will be**

**यदि T : $R^2$ (R) → $R^2$ (R) जो कि T(2, 3) = (4, 5) तथा T (1, 0) = (0, 0), से परिभाषित है, तो T (x, y) का मान होगा**

(a) $\left(\frac{4y}{3}, \frac{5y}{3}\right)$ (b) $\left(\frac{5y}{3}, \frac{4y}{3}\right)$

(c) $\left(x - \frac{4y}{3}, \frac{5y}{3}\right)$ (d) $\left(\frac{3y-4y}{3}, \frac{3y-x}{3}\right)$

**Ans. (a) :** We have that {(2, 3), (1, 0)} is a basis of $R^2$(R) and hence for every (x, y)∈$R^2$ (R)

(x, y) = α (2, 3) + β (1, 0)

for some scales α, β∈ R which gives

$$\alpha = \frac{y}{3} \ \& \ \beta = x - \frac{2y}{3}$$

Now if a linear transformation $T: R^2 \to R^2$ is defined by T (2, 3) = (4, 5) and T (1, 0) = (0, 0)

then we must have

T (x, y) = α T (2, 3) + β T (1, 0)

$\Rightarrow \quad T(x, y) = \alpha(4,5) + \beta(0,0)$

$\Rightarrow \quad T(x, y) = (4\alpha, 5\alpha) = \left(\frac{4y}{3}, \frac{5y}{3}\right)$

**29. The maximum number of normals, which can be drawn from a given point to the paraboloid, $ax^2 + by^2 = 2cz$ is**

**एक दिये हुए बिन्दु से परवलयज $ax^2 + by^2 = 2cz$ पर अधिकतम कितने अभिलंब खींचे जा सकते है?**

(a) 3 (b) 4

(c) 5 (d) 6

**Ans. (c) :** Given paraboloid is $ax^2 + by^2 = 2cz$ .....(i)

The equation of normal to (i) at (α, β, γ) is

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{-c} \quad ....(ii)$$

If it passes through point ($x_1$ ,$y_1$ ,$z_1$) then

$$\frac{x_1-\alpha}{a\alpha} = \frac{y_1-\beta}{b\beta} = \frac{z_1-\gamma}{-c} = r \quad \text{(Say)}$$

$$\Rightarrow \alpha = \frac{x_1}{1+ar}, \quad \beta = \frac{y_1}{1+br}, \quad \gamma = z_1 + cr \quad ....(iii)$$

also (α, β, γ) lies on (i)

∴ $a\alpha^2 + b\beta^2 = 2c\gamma$

Putting values of α, β, γ from (iii), we have

$$\Rightarrow \frac{a x_1^2}{(1+ar)^2} + \frac{by_1^2}{(1+br)^2} = 2c(z_1 + cr)$$

which is a fifth degree equation in r. Hence there can be atmost five normals from a given point to (i).

**30. Which one of the following is NOT a topology on X = {a, b, c}?**

**निम्नलिखित में से कौन X = {a, b, c} पर संस्थिति नहीं है?**

(a) {ϕ,{a},{b,c},X}

(b) {ϕ,{b},{a,c},X}

(c) {ϕ,{a},{b},{a,b},X}

(d) {ϕ,{a},{b},{c},{a,b},X}

**Ans. (d) : Definition:** Let X be a set and τ be the collection of subsets of X. The τ is called a topology on X if-

(i) ϕ And X are in τ.

(ii) The arbitrary union of elements of τ are also in τ.

(iii) The finite intersection of elements of τ are also in τ.

Verify that options (a), (b) and (c) satisfies the all above conditions.

In option (d) let τ = {ϕ, {a}, {b}, {c}, {a, b}, X}

As {b}, {c} ∈ τ

but {b} ∪ {c} = {b, c} ∉ τ

∴ τ is not a topology on X = {a, b, c}.

**31. The plane ax + by + cz = 0 cuts the cone yz + zx +xy = 0 in perpendicular lines, if**

**समतल ax + by + cz = 0 शंकु yz + zx +xy = 0 का लम्बवत् रेखाओं में काटेगा, यदि**

(a) $a+b+c=0$ (b) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

(c) $a^2+b^2+c^2=0$ (d) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=0$

**Ans. (b) :**

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be line of intersection of plane and cone then $mn + nl + lm = 0$ and $al + bm + cn = 0$

$$\Rightarrow (m+l)\left(-\frac{(al+bm)}{c}\right)+lm=0$$

$$\Rightarrow a\left(\frac{l}{m}\right)^2+(a+b-c)\left(\frac{l}{m}\right)+b=0$$

Let its roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$

Then sum of roots $\frac{l_1}{m_1}+\frac{l_2}{m_2}=\frac{-(a+b-c)}{a}$

and product of roots $\frac{l_1}{m_1}\times\frac{l_2}{m_2}=\frac{b}{a}$

or $\frac{l_1l_2}{\frac{1}{a}}=\frac{m_1m_2}{\frac{1}{b}}$

Similarly $\frac{m_1m_2}{\frac{1}{b}}=\frac{n_1n_2}{\frac{1}{c}}$

$$\Rightarrow \frac{l_1l_2}{1/a}=\frac{m_1m_2}{1/b}=\frac{n_1n_2}{1/c}$$

Since the plane cuts the cone in perpendicular lines if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

which is only possible when $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

**32. If for the functional $J[y]=\int_a^b F(x,y,y')dx$, the integrand does not depend on x, then the first integral of Euler's equation is**

**फलनक $J[y]=\int_a^b F(x,y,y')dx$, के लिये यदि समाकल्य का मान x पर निर्भर नहीं करता हे, तो आइलर के समीकरण का प्रथम समाकल होगा**

(a) $F_y = C$ (b) $F - y' F_y = C$

(c) $F - y' F_y = C$ (d) $F_y - \frac{d}{dx}F_{y'} = 0$

**Ans. (b) :** Given Functional is $J[y] = \int_a^b F(x,y,y')dx$ in which F (x, y, y') does not depend on x i.e. $\frac{\partial F}{\partial x}=0$

So by Euler's equation

$$\frac{d}{dx}\left[F-y'\frac{\partial F}{\partial y'}\right]-\frac{\partial F}{\partial x}=0 \text{ gives}$$

$$\frac{d}{dx}\left[F-y'\frac{\partial F}{\partial y'}\right]-0=0$$

$$\Rightarrow F-y'\frac{\partial F}{\partial y'}=C \text{ (Constant)}$$

**33.** $\lim\limits_{(x,y)\to(0,0)}\frac{x^3+y^3}{x-y}$

(a) Exists and equals to zero/का अस्तित्व है और शून्य के बराबर है

(b) Exists but not equal to zero/का अस्तित्व है परन्तु शून्य के बराबर नहीं है

(c) Exists but not unique/का अस्तित्व है परन्तु अद्वितीय नहीं है

(d) Does not exist/का अस्तित्व नहीं है

**Ans. (d) :** Take y=0. Then we have

$$\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x-y}=\lim_{x\to0}x^2=0$$

Take y = sin x. Then we have

$$\lim_{x\to0}\frac{x^3+\sin^3x}{x-\sin x}$$
$$=\lim_{x\to0}\frac{3x^2+3\sin^2x\cos x}{1-\cos x}$$
$$=\lim_{x\to0}\frac{6x+6\sin x\cos^2x-3\sin^3x}{\sin x}$$
$$=\lim_{x\to0}\left(\frac{6x}{\sin x}+6\cos^2x-3\sin^2x\right)$$
$$=12$$

Since limits are different on different paths to the origin, the conclusion is that limit does not exist.

**34.** $L^{-1}\left(\frac{s-1}{s^2-6s+13}\right)$ **is equal to**

$L^{-1}\left(\frac{s-1}{s^2-6s+13}\right)$ **बराबर है**

(a) $e^{3x}\sin 2x$

(b) $e^{2x}\cos 2x$

(c) $e^{2x}(\sin 3x+\cos 3x)$

(d) $e^{3x}(\sin 2x+\cos 2x)$

**Ans. (d) :**

$$L^{-1}\left\{\frac{s-1}{s^2-6s+13}\right\}=L^{-1}\left\{\frac{s-3+2}{(s-3)^2+4}\right\}$$

$$=L^{-1}\left\{\frac{(s-3)}{(s-3)^2+2^2}\right\}+L^{-1}\left\{\frac{2}{(s-3)^2+2^2}\right\}$$

$= e^{3x}\cos 2x + e^{3x}\sin 2x$

$= e^{3x}(\cos 2x + \sin 2x)$

**35. Let X = {a, b, c} and T = {$\phi$, {a},{a, b}, X} be a topology on X. Then (X, T) is**

**माना X = {a, b, c} और T = {$\phi$, {a},{a, b}, X} X पर एक संस्थिति है। तब (X, T) है**

(a) Compact only/केवल संहत
(b) Connected only/केवल सम्बद्ध
(c) Both compact and connected/संहत एवं सम्बद्ध दोनों
(d) Neither compact nor connected/न तो संहत न ही सम्बद्ध

**Ans. (c) :**
Given X = {a, b, c} and T = {ϕ, {a}, {a, b}, X}
Topological space (X, T) is Connected if and only if non-empty subset of X which is both open and Closed in X is X itself. In T there is no non-empty proper subset of X which is both open and closed therefore (X, T) is Connected. Since X is a finite set, so it is compact.

**36. The Fourier transform of $e^{-\frac{x^2}{2}}$ is**
**$e^{-\frac{x^2}{2}}$ का फोरियर रूपांतरण है**

(a) $\sqrt{2\pi}e^{-\frac{s^2}{2}}$ (b) $\sqrt{2}\ \pi e^{\frac{s^2}{2}}$
(c) $\sqrt{2\pi}\ e^{\frac{s^2}{2}}$ (d) $\sqrt{2}\ \pi e^{-\frac{s^2}{2}}$

**Ans. (a) :** Let F (s) be Fourier transform of f(x) = $e^{\frac{-x^2}{2}}$ then

$$F(s) = \int_{-\infty}^{\infty} f(x)\ e^{-isx}dx$$
$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\ e^{-isx}dx$$
$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}-isx}\ dx$$
$$= \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}}+\frac{is}{\sqrt{2}}\right)^2-\frac{s^2}{2}}dx$$
$$= e^{-\frac{s^2}{2}}\int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}}+\frac{is}{\sqrt{2}}\right)^2}dx$$

Let $\frac{x}{\sqrt{2}}+\frac{is}{\sqrt{2}} = y \Rightarrow \frac{dx}{\sqrt{2}} = dy \Rightarrow dx = \sqrt{2}\ dy$
$x \to -\infty$ then $y \to -\infty$ and $x \to \infty$ then $y \to \infty$

$$\Rightarrow F(s) = \sqrt{2}\,e^{-\frac{s^2}{2}\int_{-\infty}^{\infty}e^{-y^2}dy} = \sqrt{2}\,e^{-\frac{s^2}{2}} \times 2\int_{-\infty}^{\infty} e^{-y^2}dy$$

Let $y^2 = z \Rightarrow 2y\ dy = dz \Rightarrow dy = \frac{dz}{2y} = \frac{dz}{2z^{1/2}}$
As $y \to 0$ then $z \to 0$ and $y \to \infty$ then $z \to \infty$

$$= \frac{2}{2}\sqrt{2}\,e^{-\frac{s^2}{2}}\int_0^{\infty} e^{-z}z^{-\frac{1}{2}}\,dz = \sqrt{2}\,e^{-\frac{s^2}{2}} \times \sqrt{\pi}$$
$$= \sqrt{2\pi}\ e^{-\frac{s^2}{2}}$$

**37. A subspace of a normal space is**
**प्रसामान्य समष्टि की उपसमष्टि होगी**

(a) Normal/प्रसामान्य
(b) Hausdorff/हाऊसडार्फ
(c) Need not normal/प्रसामान्य होना जरूरी नहीं
(d) Closed/संवृत

**Ans. (c) :** A normal space is topological space where any two disjoint closed sets can be separated by disjoint open neighborhoods. However, this property does not always transfer to subspaces. For example, while  with the standard topology is normal, certain subspaces of it (with their subspace topology) may fail to be normal.
This demonstrates that being normal is not hereditary. (i.e., the property is not guaranteed for subspaces).

**38. The second order partial differential equation $\frac{\partial^2 z}{\partial x^2} - 2\sin x\frac{\partial^2 z}{\partial x\partial y} - \cos^2 x\frac{\partial^2 z}{\partial y^2} - \cos x\frac{\partial z}{\partial y} = 0$ is**
**द्वितीय क्रम की आंशिक अवकल समीकरण $\frac{\partial^2 z}{\partial x^2} - 2\sin x\frac{\partial^2 z}{\partial x\partial y} - \cos^2 x\frac{\partial^2 z}{\partial y^2} - \cos x\frac{\partial z}{\partial y} = 0$ है**

(a) Parabolic for all values of x, y/सभी x, y के मानों के लिये परवलयिक
(b) Parabolic in half plane $x \geq 0$/अर्ध तल $x \geq 0$ में परवलयिक
(c) Parabolic in the half plane $y \geq 0$/अर्ध तल $y \geq 0$ में परवलयिक
(d) Hyperbolic for all values of x, y/सभी x, y के मानों के लिये अतिपरवलयिक

**Ans. (d) :** Given P.D.E. is
$$\frac{\partial^2 z}{\partial x^2} - 2\sin x\ \frac{\partial^2 z}{\partial x\partial y} - \cos^2 x\ \frac{\partial^2 z}{\partial y^2} - \cos x\ \frac{\partial z}{\partial y} = 0$$
$b^2 - 4ac$
$b = -2\sin x$, $a = 1$, $c = -\cos^2 x$
Consider $(-2\sin x)^2 - 4(1)(-\cos^2 x)$
$= 4\sin^2 x + 4\cos^2 x$
$= 4 > 0$
⇒ Given P.D.E. is hyperbolic for all values of x, y.

**39. Let V = {(a, b, c, d): b – 2c + d = 0} be a subspace of $R^4$. Then dimension of V is**
**माना V = {(a, b, c, d): b – 2c + d = 0} की एक उपसमष्टि है। तो V की विमा है**

(a) 0 (b) 1
(c) 2 (d) 3

**Ans. (d) :** Given V = {(a, b, c, d) : b – 2c + d = 0} Subspace of $R^4$
⇒ V = {(a, 2c – d, c, d) : a, c, d ∈ R}
Showing that there are 3 arbitrary entries
⇒ dim V = 3

**40. Let $f_1(x) = 4$, $f_2(x) = x^3$ and $f_3(x) = 1 + Ax + Bx^2$. If $f_3(x)$ is orthogonal to $f_1(x)$ and $f_2(x)$ on the interval (–2, 2). Then**
**माना $f_1(x) = 4$, $f_2(x) = x^3$ तथा $f_3(x) = 1 + Ax + Bx^2$ यदि $f_3(x)$ अंतराल (–2, 2) पर $f_1(x)$ और $f_2(x)$ पर लांबिक हो, तो**

(a) A = 0, B = 1 (b) A = 0, B = 0
(c) A = 0, B = $\frac{3}{4}$ (d) A = 0, B = $-\frac{3}{4}$

**Ans. (d) :**
Given $f_1(x) = 4$, $f_2(x) = x^3$ and $f_3(x) = 1+Ax+Bx^2$
Also $f_3(x)$ is orthogonal to $f_1(x)$ and $f_2(x)$ on the interval $(-2, 2)$

$$\Rightarrow \int_{-2}^{2} f_3(x)\ f_1(x)\ dx = 0$$

$$\Rightarrow \int_{-2}^{2}\left(1+Ax+Bx^2\right)(4)\ dx = 0$$

$$\Rightarrow 4\left[x+A\frac{x^2}{2}+B\frac{x^3}{3}\right]_{-2}^{2} = 0$$

$$\Rightarrow 4\left[\left(2+2A+\frac{8}{3}B\right)-\left(-2+2A-\frac{8}{3}B\right)\right] = 0$$

$$\Rightarrow 4+\frac{16B}{3}=0 \quad \Rightarrow B=-\frac{12}{16}$$

$$\Rightarrow \boxed{B=-\frac{3}{4}}$$

and $\int_{-2}^{2} f_3(x)\ f_2(x)\ dx = 0$

$$\Rightarrow \int_{-2}^{2}\left(1+Ax+Bx^2\right)\left(x^3\right)dx = 0$$

$$\Rightarrow \int_{-2}^{2}\left(x^3+Ax^4+Bx^5\right)dx = 0$$

$$\Rightarrow \left[\frac{x^4}{4}+A\frac{x^5}{5}+B\frac{x^6}{6}\right]_{-2}^{2} = 0$$

$$\Rightarrow \left[4+\frac{32}{5}A+\frac{32}{3}B\right]-\left(4-\frac{32}{5}A+\frac{32}{3}B\right) = 0$$

$$\frac{64}{5}A = 0 \quad \Rightarrow \boxed{A=0}$$

**41. The relativistic form of Newton's second law of motion is**
**न्यूटन के गति के दूसरे नियम का सापेक्षकीय रूप है**

(a) $F=\frac{mc^2}{c^2-v^2}\frac{dv}{dt}$

(b) $F=\frac{m\sqrt{c^2-v^2}}{c}\frac{dv}{dt}$

(c) $F=\frac{mc}{\sqrt{c^2-v^2}}\frac{dv}{dt}$

(d) $F=\frac{m\left(c^2-v^2\right)}{c^2}\frac{dv}{dt}$

**Ans. (c) :** The relativistic form of Newton's second law of motion is given as $F = \frac{mc}{\sqrt{c^2-v^2}}\frac{dv}{dt}$ where m = mass, c = velocity of light, v = velocity of mass m.

**42. In the Laurent series expansion of the function $f(z) = \frac{1}{z-1}-\frac{1}{z-2}$, valid in the region $|z| > 2$, the coefficient of $\frac{1}{z^2}$ is**

**फलन $f(z) = \frac{1}{z-1}-\frac{1}{z-2}$, के लाराँ श्रेणी प्रसार में, जो क्षेत्र $|z| > 2$ में मान्य है, $\frac{1}{z^2}$ का गुणांक है**

(a) 0 (b) 1
(c) – 1 (d) 2

**Ans. (c) :** Given $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$, $|z| > 2$ ...(i)

f(z) can also be written as

$$f(z) = \frac{1}{z\left(1-\frac{1}{z}\right)}-\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$\Rightarrow f(z) = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} \quad ....(ii)$$

$$\because\ |z|>2 \Rightarrow \left|\frac{1}{z}\right|<\frac{1}{2}<1$$

$$\text{and } \left|\frac{2}{z}\right|<1$$

$$\Rightarrow f(z) = \frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+......\right) - \frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+......\right)$$

$$\Rightarrow f(z) = \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+....\right) - \left(\frac{1}{z}+\frac{2}{z^2}+\frac{(2)^2}{z^3}+\frac{(2)^3}{z^4}+....\right)$$

$$\Rightarrow f(z) = -\frac{1}{z^2}-\frac{3}{z^3}-\frac{7}{z^4}-......$$

$\therefore$ Coefficient of $\frac{1}{z^2}$ is $-1$

**43. Which of the following is a generator of the cone $2x^2-3y^2+4z^2=0$?**

**निम्नलिखित में से कौन शंकु $2x^2-3y^2+4z^2=0$ का एक जनक है?**

(a) $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ (b) $\frac{x}{1}=\frac{y}{\sqrt{2}}=\frac{z}{1}$

(c) $\frac{x}{2}=\frac{y}{1}=\frac{z}{-1}$ (d) $\frac{x}{-1}=\frac{y}{2}=\frac{z}{1}$

**Ans. (b) :** The direction cosines of generator of a cone satisfies the equation of the cone. In option (b) $\frac{x}{1}=\frac{y}{\sqrt{2}}=\frac{z}{1}$ direction cosines of generators are 1, $\sqrt{2}$, 1 which satisfy $2x^2 - 3y^2 + 4z^2 = 0$ i.e. $2(1)^2 - 3\left(\sqrt{2}\right)^2+4(1)^2=0$ Therefore $\frac{x}{1}=\frac{y}{\sqrt{2}}=\frac{z}{1}$ is a generator of the cone.

**44. If all vertices are of degree at least 2, then the largest number of vertices in a graph with 35 edges is/यदि सभी शीर्ष कम से कम 2 घात के हो, तो 35 किनारों वाले ग्राफ में शीषों की अधिकतम संख्या होगी**

(a) 25 (b) 23
(c) 21 (d) 27

**Ans. (*) :** We have

2 (number of edges) = sum of degree of vertex

and because each vertex of graph is of at least degree 2 with 35 edges we get

2 (35) = sum of degree of vertex

$\geq 2+2+....+2$

$= 2n$

which gives $n \leq 35$

**45. The spheres $x^2+y^2+z^2=25$ and $x^2+y^2+z^2-24x-40y-18z+225=0$**
**गोले $x^2+y^2+z^2=25$ तथा $x^2+y^2+z^2-24x-40y-18z+225=0$**

(a) Do not intersect/प्रतिच्छेद नहीं करते है
(b) Intersect in two points/दो बिन्दुओं पर प्रतिच्छेद करते है
(c) Touch internally/अन्तः स्पर्श करते है
(d) Touch externally/बाह्यतः स्पर्श करते है

**Ans. (d) :** Given spheres can be written as

$(x-0)^2+(y-0)^2+(z-0)^2=(5)^2$ and ....(i)

$(x-12)^2+(y-20)^2+(z-9)^2=(20)^2$ ....(ii)

Sphere (i) has centre $C_1 = (0, 0, 0)$ and radius $r_1 = 5$

and Sphere (ii) has centre $C_2 = (12, 20, 9)$ and radius $r_2 = 20$

$\therefore$ Distance between both the centres

$$C_1C_2 = \sqrt{(12-0)^2+(20-0)^2+(9-0)^2}$$

$$= \sqrt{144+400+81}$$

$$= \sqrt{625}$$

$$= 25$$

and sum of both the radius $r_1 + r_2 = 25$

$\Rightarrow C_1C_2 = r_1 + r_2$

$\Rightarrow$ both the spheres touch externally.

**46. If A and B are measurable subsets of interval [a, b], then which of the following is correct?**
**यदि A और B अंतराल [a, b], के मेय उपसमुच्चय हो, तो निम्नलिखित में कौन सही है**

(a) $A \cup B$ is not measurable/मेय $A \cup B$ नहीं है
(b) $A \cap B$ is not measurable/मेय $A \cap B$ नहीं है
(c) $A \cup B$ and $A \cap B$ both are measurable/ $A \cup B$ तथा $A \cap B$ दोनों मेय है
(d) Neither $A \cup B$ nor $A \cap B$ are measurable/न तो $A \cup B$ न ही $A \cap B$ मेय है

**Ans. (c) : I.** A countable union of measurable sets is measurable.

**II.** A countable intersection of measurable sets is measurable.

**47. Solution of the integral equation $y(x) = x + \int_0^x \sin(x-t)y(t)\,dt$ is**
**समाकल समीकरण $y(x) = x + \int_0^x \sin(x-t)y(t)\,dt$ का हल है**

(a) $y(x) = x + \frac{x^2}{3}$

(b) $y(x) = x - \frac{x^3}{6}$

(c) $y(x) = x + \frac{x^3}{6}$

(d) $y(x) = x^2 + x^3$

**Ans. (c) :** $y(x) = x + \int_0^x \sin(x-t)\,y(t)dt$

$L[y(x)] = L[x + \sin x\, y(x)]$ (by Convolution theorem)

$L[y(x)] = L[x] + L[\sin x]\, L[y(x)]$

$\Rightarrow L[y(x)]\{1 - L[\sin x]\} = L[x]$

$\Rightarrow L[y(x)] = \frac{L[x]}{1 - L[\sin x]}$

$\Rightarrow L[y(x)] = \frac{1/s^2}{1 - \frac{1}{1+s^2}}$

$\Rightarrow L[y(x)] = \frac{1/s^2}{s^2/1+s^2}$

$\Rightarrow L\ [y(x)] = \frac{1+s^2}{s^4}$

$\Rightarrow L\ [y(x)] = \frac{1}{s^4} + \frac{1}{s^2}$

$\Rightarrow y(x) = L^{-1}\left(\frac{1}{s^4}\right) + L^{-1}\left(\frac{1}{s^2}\right)$

$\Rightarrow y(x) = \frac{1}{3!}\ x^3 + x$

$\Rightarrow y(x) = x + \frac{x^3}{6}$

**48. If L (F(t)) = f (s), then L ($t^n$F(t)) is equal to**
**यदि L (F(t)) = f (s), तो L ($t^n$F(t)) का मान है**

(a) $(-1)^n \frac{d^n}{ds^n}(f(s))$ (b) $\frac{d^n}{ds^n}(f(s))$

(c) $(-1)^{n-1} \frac{d^n}{ds^n}(f(s))$ (d) $(-1)^{n+1} \frac{d^n}{ds^n}(f(s))$